Betting Schemes for Assessing Coherent Numerical and Comparative Conditional Possibilities

Davide Petturiti DAVIDE.PETTURITI@UNIPG.IT
Dept. Economics, University of Perugia, Italy

Barbara Vantaggi BARBARA.VANTAGGI@UNIROMA1.IT
Dept. MEMOTEF, "La Sapienza" University of Rome, Italy

Abstract

We introduce coherence conditions having a betting scheme interpretation both for a numerical and a comparative conditional possibility assessment. The conditional bets are considered under partially resolving uncertainty and assuming consonance. This means that we allow situations in which the agent may only acquire the information that a non-impossible event occurs, without knowing which is the true state of the world. Further, he/she can only consider families of nested non-impossible events in computing the gain and has a systematically optimistic behavior. Both conditions are proved to be equivalent to the existence of a conditional possibility agreeing with an axiomatic definition based on the algebraic product t-norm, that extends, either numerically or comparatively (through the induced comparative conditional possibility relation), the given assessment.

Keywords: Conditional possibility, betting scheme, comparative conditional possibility, coherence

1. Introduction

Since their introduction by Zadeh [34], possibility measures have gathered a lot of attention, mainly due to their key role in fuzzy set theory and soft computing.

Possibility theory has been substantially developed by Dubois and Prade [16] and is nowadays a well-established uncertainty framework which is normally seen as an alternative to probability theory. Actually there are several links between the two frameworks: for example, possibility measures can be characterized as upper envelopes of the class of all de Finetti's coherent extensions of a probability measure under suitable logical conditions (see [17, 8, 9]). Other probabilistic interpretations of possibility measures exist, such as the one in terms of random sets [27], the one in terms of large deviations [28], or the one in terms of likelihood functions [19, 3].

Further, possibility theory can be reformulated inside the Dempster-Shafer theory of evidence [13, 30], through the notion of *consonance*, or inside the more general theory of coherent lower/upper probabilities [32].

The concept of conditioning for possibility measures has been largely debated in the related literature. Various definitions of conditional possibility have been proposed (see, for instance, [16, 11, 33]), mainly in analogy with probability theory or by means of suitable rules of conditioning. Another alternative is to interpret a conditional possibility as an upper conditional probability, i.e., as the upper envelope of a particular class of conditional probabilities [18, 9].

Here, we adopt a different approach in which a conditional possibility is axiomatically defined as a primitive concept [1]. In this definition $\Pi(\cdot|\cdot)$ is a function, whose domain is a structured set of conditional events, so that $\Pi(E|H)$ can be defined for any pair of events (E,H), with $H \neq \emptyset$. Actually, there is an entire class of definitions, referred to as T*-conditional possibility*, parametrized by the choice of the t-norm T, which is the operation expressing the link among $\Pi(A|B)$, $\Pi(B|C)$ and $\Pi(A|C)$, when $A \subseteq B \subseteq C$. The theory of T-conditional possibility has been developed in a series of papers [21, 4, 7, 2], where a notion of coherence for a partial assessment has been introduced. Such notion of coherence is expressed in terms of extendibility of a partial assessment to a T-conditional possibility on a structured domain.

In this paper we focus on T-conditional possibilities, where T is the algebraic product t-norm, that we simply call *conditional possibilities*. This axiomatic definition is a generalization of Dempster's rule of conditioning for plausibility functions [13].

We first consider a partial numerical assessment given on an arbitrary set of conditional events and propose a coherence condition having a betting scheme interpretation (namely, *b-coherence*). Such condition relies on the principle of *partially resolving uncertainty* introduced by Jaffray in [25]. Adopting such principle, we allow that an agent may only acquire the information that an event $B \neq \emptyset$ occurs, without knowing which is the true state of the world $\omega \in B$. This amounts to consider a subset of $\mathscr{U} = \mathscr{P}(\Omega) \setminus \{\emptyset\}$ as the domain of the gain function in a combination of bets. Further, we assume *consonance* meaning that the agent in his/her mental speculation on the gain in a combination of bets is allowed only to consider subfamilies of $\mathscr{U}$ that can be increasingly ordered by set inclusion (namely, chains). Moreover, we assume a systematically optimistic behavior by taking $\mathbf{1}_A^{\mathbf{U}}(B) = \max_{\omega\in B} \mathbf{1}_A(\omega)$ as the reward for betting on A when we acquire the information that $B \neq \emptyset$ occurs. Here, $\mathbf{1}_A$ is the indicator of event A, while $\mathbf{1}_A^{\mathbf{U}}$ is referred to as *upper generalized indicator*. Indeed, assuming a linear utility scale, $\mathbf{1}_A$ can be seen as a gamble paying 1 monetary unit on every state of the world $\omega \in A$ and 0 otherwise. Working under partially resolving uncertainty, that is considering a gain defined on events $B \neq \emptyset$, the agent is optimistic since betting on A he/she thinks to receive 1 monetary unit whenever $B \cap A \neq \emptyset$ and 0 only when $B \cap A = \emptyset$.

The b-coherence condition requires that for every finite subfamily of conditional events we can find a chain whose top element is the union of the involved conditioning events. Moreover, for every possible combination of bets, the resulting gain is asked not to be uniformly negative over the chain. We show that b-coherence is equivalent to the existence of a conditional possibility extending the given assessment (namely, *e-coherence*).

Then, we consider a comparative assessment, expressed by a pair of binary relations $(\precsim, \prec)$ comparing conditional events under the same conditioning event, in terms of their conditional possibility. Also in this case we propose a coherence condition having a betting scheme interpretation (namely, *bc-coherence*). Again we assume partially resolving uncertainty and consonance, as well as a systematically optimistic behavior. The bc-coherence condition requires the existence of fixed positive "penalty fees" to pay in case of bets on strict comparisons. We require that, for every finite subfamily of comparisons, we can find a chain whose top element is the union of the involved conditioning events. Moreover, for every possible combination of bets, the resulting gain is asked not to be uniformly negative over the chain. We show that bc-coherence is equivalent to the existence of a conditional possibility inducing a comparative conditional possibility relation that extends the given assessment (namely, *ec-coherence*).

Both b-coherence and bc-coherence conditions provide an operational tool to elicit subjective numerical or comparative conditional possibility assessments. Further, we show that they reveal to be necessary and sufficient conditions for the extendibility of the given assessment to any larger domain.

The issue of proposing a definition of subjective possibility has been already faced in the literature (see, e.g., [23, 12, 20]). In particular, in [23] a betting semantics is proposed, essentially interpreting possibility assessments in the context of imprecise probabilities. We stress that our approach considers conditional possibility as a primitive concept and provides a complete characterization of coherence. The key feature of our betting schemes is that the gain G of a combination of bets is not defined on the states of the world ω's but on non-impossible events B's.

The paper is organized as follows. In Section 2 we recall the axiomatic definition of conditional possibility. Section 3 deals with coherence of a numerical conditional possibility assessment. Section 4 deals with coherence of a comparative conditional possibility assessment. Finally, in Section 5 we draw conclusions and future perspectives.

2. Axiomatically Defined Conditional Possibility

Throughout this paper we assume $\Omega = \{\omega_1, \dots, \omega_m\}$ is a finite non-empty set and denote by $\mathscr{P}(\Omega)$ its power set, while $\mathscr{P}(\Omega)^0 = \mathscr{P}(\Omega) \setminus \{\emptyset\}$.

A *possibility measure* is a function $\Pi : \mathscr{P}(\Omega) \to [0,1]$ satisfying:

(i) $\Pi(\emptyset) = 0$ and $\Pi(\Omega) = 1$;

(ii) $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$, for every $A, B \in \mathscr{P}(\Omega)$.

Every possibility measure Π is associated to a dual *necessity measure* $N : \mathscr{P}(\Omega) \to [0,1]$ defined, for every $A \in \mathscr{P}(\Omega)$, as

$$N(A) = 1 - \Pi(A^c). \tag{1}$$

Possibility/necessity measures are particular plausibility/belief functions in the Dempster-Shafer theory of evidence [13, 30]. Every possibility measure Π on $\mathscr{P}(\Omega)$ is completely characterized by the *Möbius inverse* of its dual necessity measure N, i.e., by the function $m : \mathscr{P}(\Omega) \to [0,1]$ defined, for every $A \in \mathscr{P}(\Omega)$, as

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} N(B). \tag{2}$$

Indeed, given m, for every $A \in \mathscr{P}(\Omega)$, we have that

$$N(A) = \sum_{B \subseteq A} m(A) \quad \text{and} \quad \Pi(A) = \sum_{B \cap A \neq \emptyset} m(A). \tag{3}$$

In general, the Möbius inverse of a belief function turns out to satisfy the following properties

$$m(\emptyset) = 0 \quad \text{and} \quad \sum_{A \in \mathscr{P}(\Omega)} m(A) = 1. \tag{4}$$

Hence, if we disregard $m(\emptyset) = 0$, the Möbius inverse m can be considered as a probability distribution over $\mathscr{U} = \mathscr{P}(\Omega)^0$, giving rise to a probability measure over $\mathscr{P}(\mathscr{U})$. In particular, the elements of $\mathscr{U}$ where m is strictly positive are called *focal elements*. As proved in [30, 24], a belief function is a necessity measure if and only if the corresponding Möbius inverse has nested focal elements, that is they can be linearly ordered according to set inclusion. Necessity/possibility measures are called *consonant* belief/plausibility functions in [30] for this property.

Consider $H \in \mathscr{U}$. If $H = \{\omega_{i_1}, \dots, \omega_{i_h}\}$, denote by $\mathbf{chains}(\mathscr{U}, H)$ the collection of subfamilies of $\mathscr{U}$ such that $\mathscr{D} = \{D_1, \dots, D_h\} \in \mathbf{chains}(\mathscr{U}, H)$ if and only if $D_1 = \{\omega_{i_{\sigma(1)}}\}$, $D_2 = \{\omega_{i_{\sigma(1)}}, \omega_{i_{\sigma(2)}}\}, \dots, D_h = \{\omega_{i_{\sigma(1)}}, \dots, \omega_{i_{\sigma(h)}}\} = H$, where σ is a permutation of $\{1, \dots, h\}$. Notice that the elements of $\mathbf{chains}(\mathscr{U}, H)$ are in one-to-one correspondence with permutations of elements of H.

Every event $A \in \mathscr{P}(\Omega)$ is associated to the *indicator* $\mathbf{1}_A : \Omega \to \{0,1\}$, where $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and 0 otherwise. Further, the corresponding *upper generalized indicator* is the function $\mathbf{1}_A^{\mathbf{U}} : \mathscr{U} \to \{0,1\}$ defined, for every $B \in \mathscr{U}$, as

$$\mathbf{1}_A^{\mathbf{U}}(B) = \max_{\omega \in B} \mathbf{1}_A(\omega). \tag{5}$$

We consider the following axiomatic definition of conditional possibility, introduced in [1] and further studied in [21, 7, 2] under the name of *T-conditional possibility*, where T is a t-norm. In this paper we restrict to the particular case in which T is the algebraic product.

Definition 1 *Let $\mathscr{H} \subseteq \mathscr{P}(\Omega)^0$ be an additive class, i.e., a non-empty family closed with respect to unions. A* **conditional possibility** *is a function $\Pi : \mathscr{P}(\Omega) \times \mathscr{H} \to [0,1]$ satisfying the following conditions:*

- *(i) $\Pi(E|H) = \Pi(E \cap H|H)$, for every $E \in \mathscr{P}(\Omega)$ and $H \in \mathscr{H}$;*
- *(ii) $\Pi(\cdot|H)$ is a possibility measure on $\mathscr{P}(\Omega)$, for every $H \in \mathscr{H}$;*
- *(iii) $\Pi(E \cap F|H) = \Pi(E|H) \cdot \Pi(F|E \cap H)$, for every $H, E \cap H \in \mathscr{H}$ and $E, F \in \mathscr{P}(\Omega)$.*

Further we say that a conditional possibility is *full on* $\mathscr{P}(\Omega)$ if $\mathscr{H} = \mathscr{P}(\Omega)^0$, i.e., if it is defined on the entire $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)^0$.

Every conditional possibility $\Pi(\cdot|\cdot)$ on $\mathscr{P}(\Omega) \times \mathscr{H}$ gives rise to a linearly ordered class $\{\Pi_0, \dots, \Pi_k\}$ of possibility measures on $\mathscr{P}(\Omega)$ said *$\mathscr{H}$-minimal agreeing class*, associated to a class $\{H_0^0, \dots, H_0^k\}$ of elements of $\mathscr{H}$ decreasingly ordered by set inclusion. We actually have that conditional possibilities on $\mathscr{P}(\Omega) \times \mathscr{H}$ are in one-to-one correspondence with $\mathscr{H}$-minimal agreeing classes on $\mathscr{P}(\Omega)$ (see [29]). Given a conditional possibility $\Pi(\cdot|\cdot)$ set:

- $\Pi_0(\cdot) = \Pi(\cdot|H_0^0)$ with $H_0^0 = \bigcup_{H \in \mathscr{H}} H$;
- for $\alpha > 0$, let $H_0^\alpha = \bigcup\{H \in \mathscr{H} : \Pi_\beta(H) = 0, \beta = 0, \dots, \alpha - 1\}$, if $H_0^\alpha \neq \emptyset$, then $\Pi_\alpha(\cdot) = \Pi(\cdot|H_0^\alpha)$, and the construction stops at index k such that $H_0^{k+1} = \emptyset$.

Vice versa, given $\{\Pi_0, \dots, \Pi_k\}$, for every $E|H \in \mathscr{P}(\Omega) \times \mathscr{H}$, denoting by α_H the minimum index in $\{0, \dots, k\}$ such that $\Pi_{\alpha_H}(H) > 0$, it holds that

$$\Pi(E|H) = \frac{\Pi_{\alpha_H}(E \cap H)}{\Pi_{\alpha_H}(H)}. \tag{6}$$

The notion of $\mathscr{H}$-minimal agreeing class is analogous to that of agreeing class of probabilities (and to the ensuing notion of *zero-layers*) introduced by Coletti and Scozzafava for conditional probabilities [3].

Remark 2 *The notion of $\mathscr{H}$-minimal agreeing class does not coincide, in general, with that of $\mathscr{H}$-reduced T-nested class introduced in [2] by generalizing [7]. The essential difference in between the two concepts is that more possibility measures are needed to represent a T-conditional possibility $\Pi(\cdot|\cdot)$ if T is not a strictly monotone t-norm. This happens since the equation $\Pi_\alpha(E \cap H) = T(\Pi(E|H), \Pi_\alpha(H))$ may not have a unique solution, even if $\Pi_\alpha(H) > 0$.*

Every $\mathscr{H}$-minimal agreeing class of possibility measures $\{\Pi_0, \dots, \Pi_k\}$ on $\mathscr{P}(\Omega)$ gives rise to an linearly ordered class $\{m_0, \dots, m_k\}$ of Möbius inverses of the dual necessity measures. In particular, since every Π_α satisfies $\Pi_\alpha(H_0^\alpha) = 1$ and $\Pi_\alpha((H_0^\alpha)^c) = 0$, we have that for every Π_α there exists a (possibly non-unique) family $\mathscr{D} \in \mathbf{chains}(\mathscr{U}, H_0^\alpha)$ containing the focal elements of m_α. This implies that, for every $A \in \mathscr{P}(\Omega)$, it holds that

$$\Pi_\alpha(A) = \sum_{B \in \mathscr{U}} \mathbf{1}_A^{\mathbf{U}}(B) \cdot m_\alpha(B) = \sum_{D \in \mathscr{D}} \mathbf{1}_A^{\mathbf{U}}(D) \cdot m_\alpha(D), \tag{7}$$

where the first equality has been established in [31] (see also [22]). The previous expression holds, in particular, for an unconditional possibility $\Pi(\cdot)$ on $\mathscr{P}(\Omega)$, that can be regarded as a conditional possibility defined on $\mathscr{P}(\Omega) \times \{\Omega\}$, for which $\Pi(\cdot) = \Pi(\cdot|\Omega) = \Pi_0(\cdot)$.

3. Numerical Conditional Possibility Assessments

We consider a finite non-empty set $\mathscr{G} \subseteq \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)^0$ of conditional events together with a numerical conditional possibility assessment $\Pi : \mathscr{G} \to [0,1]$. Further, we denote by $\mathscr{E}(\mathscr{G}) = \{H \in \mathscr{P}(\Omega)^0 : E|H \in \mathscr{G}\}$.

The following condition expresses the coherence of an assessment Π in terms of its extendibility to a conditional possibility.

Definition 3 *An assessment $\Pi : \mathscr{G} \to [0,1]$ is said to be* **e-coherent** *if there exists a conditional possibility $\Pi' : \mathscr{P}(\Omega) \times \mathscr{H} \to [0,1]$, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions, such that $\Pi'_{|\mathscr{G}} = \Pi$.*

The following condition expresses the coherence of an assessment Π through a suitable form of betting scheme.

Definition 4 *An assessment $\Pi : \mathscr{G} \to [0,1]$ is said to be* **b-coherent** *if for every finite $\mathscr{F} = \{E_1|H_1, \dots, E_n|H_n\} \subseteq \mathscr{G}$, there exists $\mathscr{D} \in$ **chains**$(\mathscr{U}, H_0)$ with $H_0 = \bigcup_{i=1}^n H_i$, such* *that for every $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$, the function $G : \mathscr{D} \to \mathbb{R}$ defined, for every $D \in \mathscr{D}$, as*

$$G(D) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D) - \Pi(E_i|H_i)\cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D)\right] \qquad (8)$$

satisfies $\max_{D\in\mathscr{D}} G(D) \geq 0$.

The function G introduced in Definition 4 can be regarded as the gain in a combination of conditional bets assuming partially resolving uncertainty and consonance. Adopting the partially resolving uncertainty principle [25], we allow that an agent may only acquire the information that an event $B \neq \emptyset$ occurs, without knowing which is the true state of the world $\omega \in B$. This amounts to consider a subset of $\mathscr{U}$ as the domain of the gain function in a combination of bets. Further, consonance means that the agent in his/her mental speculation on the gain in a combination of bets is allowed only to consider chains whose top element is H_0. Moreover, we assume a systematically optimistic behavior since the upper generalized indicator $\mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D)$ returns 1 monetary unit whenever $D\cap E_i\cap H_i \neq \emptyset$ and 0 only when $D\cap E_i \cap H_i = \emptyset$. An analogous remark holds for $\mathbf{1}^{\mathbf{U}}_{H_i}(D)$. We have that the single bet on $E_i|H_i$ has stake λ_i and unit amount $\Pi(E_i|H_i)$. The bet is called off if no information on the occurrence of H_i is obtained in the considered chain. The b-coherence condition requires that there exists a chain $\mathscr{D}$ such that for every combination of bets on the considered conditional events the gain is not uniformly negative over $\mathscr{D}$ (no sure loss).

Theorem 5 *For a numerical conditional possibility assessment $\Pi : \mathscr{G} \to [0,1]$, the following statements are equivalent:*

(i) *Π is e-coherent;*

(ii) *Π is b-coherent.*

Proof *(i) $\Longrightarrow$ (ii).* If Π is e-coherent, then there is a conditional possibility on $\Pi' : \mathscr{P}(\Omega)\times\mathscr{H} \to [0,1]$ extending Π, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions. For every finite $\mathscr{F} = \{E_1|H_1,\ldots,E_n|H_n\} \subseteq \mathscr{G}$, let $H_0 = \bigcup_{i=1}^n H_i$. Then $\Pi(\cdot|H_0)$ is a possibility measure on $\mathscr{P}(\Omega)$ whose dual necessity measure has Möbius inverse m. Since $\Pi(H_0|H_0) = 1$ and $\Pi(H_0^c|H_0) = 0$, there exists $\mathscr{D} = \{D_1,\ldots,D_h\} \in$ **chains**$(\mathscr{U}, H_0)$ containing the focal elements of m. Consider the matrix $\mathbf{A} = [a_{ij}] \in \mathbb{R}^{(n+1)\times h}$ with

$$\begin{aligned} a_{ij} &= \mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D_j) - \Pi(E_i|H_i)\cdot\mathbf{1}^{\mathbf{U}}_{H_i}(D_j),\\ a_{(n+1)j} &= \mathbf{1}^{\mathbf{U}}_{H_0}(D_j) = 1, \end{aligned}$$

and the vector $\mathbf{b} = [0,\ldots,0,1]^T \in \mathbb{R}^{(n+1)\times 1}$. Setting $\mathbf{x} = [m(D_1),\ldots,m(D_h)]^T \in \mathbb{R}^{h\times 1}$, we have that $\mathbf{x}$ is a solution of the following system

$$\mathscr{S} : \begin{cases} \mathbf{A}\mathbf{x} = \mathbf{b},\\ \mathbf{x} \geq \mathbf{0}. \end{cases}$$

By Farkas' lemma [26], the above system $\mathscr{S}$ has solution if and only if the following system has no solution

$$\mathscr{S}^* : \begin{cases} \mathbf{y}\mathbf{A} \leq \mathbf{0},\\ \mathbf{y}\mathbf{b} > 0, \end{cases}$$

where $\mathbf{y} = [\lambda_1,\ldots,\lambda_n,y_{n+1}] \in \mathbb{R}^{1\times(n+1)}$ and $\mathbf{y}\mathbf{b} = y_{n+1}$. It holds that $\mathbf{y}\mathbf{A} \in \mathbb{R}^{1\times h}$ and, for $j = 1,\ldots,h$, the jth column of constraint $\mathbf{y}\mathbf{A} \leq \mathbf{0}$ is

$$\sum_{i=1}^{n} \lambda_i \cdot [\mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D_j) - \Pi(E_i|H_i)\cdot\mathbf{1}^{\mathbf{U}}_{H_i}(D_j)] + y_{n+1} \leq 0.$$

Therefore, for every $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$ and $y_{n+1} > 0$ there must exist at least an index $j \in \{1,\ldots,h\}$ such that $(\mathbf{y}\mathbf{A})_j > 0$. Hence, the non-solvability of $\mathscr{S}^*$ is equivalent, for every $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$, to

$$\max_{j\in\{1,\ldots,h\}} G(D_j) \geq 0,$$

where we set

$$G(D_j) = \sum_{i=1}^{n} \lambda_i \cdot [\mathbf{1}^{\mathbf{U}}_{E_1\cap H_i}(D_j) - \Pi(E_i|H_i)\cdot\mathbf{1}^{\mathbf{U}}_{H_i}(D_j)].$$

(ii) $\Longrightarrow$ (i). Suppose Π is b-coherent. Let $\mathscr{H}$ be the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions, whose top element is $H_0^0 = \bigcup_{H\in\mathscr{E}(\mathscr{G})} H$. We show that b-coherence implies the existence of a $\mathscr{H}$-minimal agreeing class $\{\Pi_0,\ldots,\Pi_k\}$ on $\mathscr{P}(\Omega)$ corresponding to a conditional possibility $\Pi'(\cdot|\cdot)$ on $\mathscr{P}(\Omega)\times\mathscr{H}$ that extends Π.

Let $n_0 = \mathrm{card}\mathscr{G}$ and $\mathscr{F}_0 = \mathscr{G} = \{E_1|H_1,\ldots,E_{n_0}|H_{n_0}\}$. Proceeding as in the last part of the proof of the converse implication, we have that b-coherence implies the existence of $\mathscr{D}_0 = \{D_1,\ldots,D_{h_0}\} \in$ **chains**$(\mathscr{U}, H_0^0)$ such that the following system has no solution

$$\mathscr{S}_0^* : \begin{cases} \mathbf{y}_0\mathbf{A}_0 \leq \mathbf{0},\\ \mathbf{y}_0\mathbf{b}_0 > 0, \end{cases}$$

where $\mathbf{y}_0 = [\lambda_1,\ldots,\lambda_{n_0},y_{n_0+1}] \in \mathbb{R}^{1\times(n_0+1)}$, $\mathbf{b}_0 = [0,\ldots,0,1]^T \in \mathbb{R}^{(n_0+1)\times 1}$ and $\mathbf{A}_0 = [a_{ij}] \in \mathbb{R}^{(n_0+1)\times h_0}$ with

$$\begin{aligned} a_{ij} &= \mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D_j) - \Pi(E_i|H_i)\cdot\mathbf{1}^{\mathbf{U}}_{H_i}(D_j),\\ a_{(n_0+1)j} &= \mathbf{1}^{\mathbf{U}}_{H_0^0}(D_j) = 1. \end{aligned}$$

In turn, by Farkas' lemma, the non-solvability of $\mathscr{S}_0^*$ is equivalent to the solvability of the following system

$$\mathscr{S}_0 : \begin{cases} \mathbf{A}_0\mathbf{x}_0 = \mathbf{b}_0,\\ \mathbf{x}_0 \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x}_0 = [x_1,\ldots,x_{h_0}]^T \in \mathbb{R}^{h_0\times 1}$. Defining $m_0 : \mathscr{P}(\Omega) \to [0,1]$ by setting $m_0(D_j) = x_j$, for $j = 1,\ldots,h_0$, and 0 otherwise, we get the Möbius inverse of a necessity measure whose dual is a possibility measure Π_0 such that $\Pi_0(H_0^0) = 1$, $\Pi_0((H_0^0)^c) = 0$, and

$$\Pi_0(E_i\cap H_i) - \Pi(E_i|H_i)\cdot\Pi_0(H_i) = 0.$$

Thus, if $\Pi_0(H_i) > 0$ we have that

$$\frac{\Pi_0(E_i \cap H_i)}{\Pi_0(H_i)} = \Pi(E_i|H_i).$$

For $\alpha > 0$, let $I_\alpha = \{i \in \{1,\dots,n_0\} : \Pi_\beta(H_i) = 0, \beta = 0,\dots,\alpha-1\}$. If $I_\alpha = \emptyset$ the construction stops, otherwise let $H_0^\alpha = \bigcup_{i \in I_\alpha} H_i$ and set $\mathscr{F}_\alpha = \{E_i|H_i\}_{i \in I_\alpha}$ where $n_\alpha = \mathrm{card}\mathscr{F}_\alpha$. Fix the enumeration $\mathscr{F}_\alpha = \{E_{k_1}|H_{k_1},\dots,E_{k_{n_\alpha}}|H_{k_{n_\alpha}}\}$. Again, we have that b-coherence implies the existence of $\mathscr{D}_\alpha = \{D_1,\dots,D_{h_\alpha}\} \in$ **chains**$(\mathscr{U}, H_0^\alpha)$ such that the following system has no solution

$$\mathscr{S}_\alpha^* : \begin{cases} \mathbf{y}_\alpha \mathbf{A}_\alpha \leq \mathbf{0}, \\ \mathbf{y}_\alpha \mathbf{b}_\alpha > 0, \end{cases}$$

where $\mathbf{y}_\alpha = [\lambda_1,\dots,\lambda_{n_\alpha}, y_{n_\alpha+1}] \in \mathbb{R}^{1\times(n_\alpha+1)}$, $\mathbf{b}_\alpha = [0,\dots,0,1]^T \in \mathbb{R}^{(n_\alpha+1)\times 1}$ and $\mathbf{A}_\alpha = [a_{ij}] \in \mathbb{R}^{(n_\alpha+1)\times h_\alpha}$ with

$$\begin{aligned} a_{ij} &= \mathbf{1}^{\mathbf{U}}_{E_{k_i}\cap H_{k_i}}(D_j) - \Pi(E_{k_i}|H_{k_i}) \cdot \mathbf{1}^{\mathbf{U}}_{H_{k_i}}(D_j), \\ a_{(n_\alpha+1)j} &= \mathbf{1}^{\mathbf{U}}_{H_0^\alpha}(D_j) = 1. \end{aligned}$$

In turn, by Farkas' lemma, the non-solvability of $\mathscr{S}_\alpha^*$ is equivalent to the solvability of the following system

$$\mathscr{S}_\alpha : \begin{cases} \mathbf{A}_\alpha \mathbf{x}_\alpha = \mathbf{b}_\alpha, \\ \mathbf{x}_\alpha \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x}_\alpha = [x_1,\dots,x_{h_\alpha}]^T \in \mathbb{R}^{h_\alpha \times 1}$. Defining $m_\alpha : \mathscr{P}(\Omega) \to [0,1]$ by setting $m_\alpha(D_j) = x_j$, for $j = 1,\dots,h_\alpha$, and 0 otherwise, we get the Möbius inverse of a necessity measure whose dual is a possibility measure Π_α such that $\Pi_\alpha(H_0^\alpha) = 1$, $\Pi_\alpha((H_0^\alpha)^c) = 0$, and

$$\Pi_\alpha(E_{k_i} \cap H_{k_i}) - \Pi(E_{k_i}|H_{k_i}) \cdot \Pi_\alpha(H_{k_i}) = 0.$$

Thus, if $\Pi_\alpha(H_{k_i}) > 0$ we have that

$$\frac{\Pi_\alpha(E_{k_i} \cap H_{k_i})}{\Pi_\alpha(H_{k_i})} = \Pi(E_{k_i}|H_{k_i}).$$

Let k be the first index such that $I_{k+1} = \emptyset$. Then $\{\Pi_0,\dots,\Pi_k\}$ is by construction a $\mathscr{H}$-minimal agreeing class corresponding to a conditional possibility $\Pi'(\cdot|\cdot)$ on $\mathscr{P}(\Omega) \times \mathscr{H}$ that extends Π. ■

By identifying every unconditional event E with the conditional event $E|\Omega$, the b-coherence condition for an unconditional assessment $\Pi : \mathscr{G} \to [0,1]$ with $\mathscr{G} \subseteq \mathscr{P}(\Omega)$ reduces to: for every finite $\mathscr{F} = \{E_1,\dots,E_n\} \subseteq \mathscr{G}$, there exists $\mathscr{D} \in$ **chains**$(\mathscr{U}, \Omega)$, such that for every $\lambda_1,\dots,\lambda_n \in \mathbb{R}$, the function $G : \mathscr{D} \to \mathbb{R}$ defined, for every $D \in \mathscr{D}$, as

$$G(D) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{E_i}(D) - \Pi(E_i)\right] \tag{9}$$

satisfies $\max_{D\in\mathscr{D}} G(D) \geq 0$. In particular, Theorem 5 guarantees that every b-coherent unconditional assessment $\Pi : \mathscr{G} \to [0,1]$ can be extended to a possibility measure $\Pi' : \mathscr{P}(\Omega) \to [0,1]$.

Example 1 *Let $\Omega = \{\omega_1,\omega_2,\omega_3\}$ and consider the assessment $\Pi(A|\Omega) = \Pi(A|B) = \Pi(B|\Omega) = \frac{1}{5}$, where $A = \{\omega_1\}$ and $B = \{\omega_1,\omega_2\}$. We show that such assessment is not b-coherent by referring to a combination of bets involving the three conditional events. At this aim take $H_0 = \Omega$, therefore* **chains**$(\mathscr{U}, H_0) = \{\mathscr{D}_1,\dots,\mathscr{D}_6\}$ *where*

$$\begin{aligned} \mathscr{D}_1 &= \{\{\omega_1\},\{\omega_1,\omega_2\},\Omega\} & \mathscr{D}_2 &= \{\{\omega_1\},\{\omega_1,\omega_3\},\Omega\} \\ \mathscr{D}_3 &= \{\{\omega_2\},\{\omega_1,\omega_2\},\Omega\} & \mathscr{D}_4 &= \{\{\omega_2\},\{\omega_2,\omega_3\},\Omega\} \\ \mathscr{D}_5 &= \{\{\omega_3\},\{\omega_1,\omega_3\},\Omega\} & \mathscr{D}_6 &= \{\{\omega_3\},\{\omega_2,\omega_3\},\Omega\} \end{aligned}$$

For $\lambda_1,\lambda_2,\lambda_3 \in \mathbb{R}$, let $\lambda_{i,j} = \lambda_i + \lambda_j$ and $\lambda_{1,2,3} = \lambda_1 + \lambda_2 + \lambda_3$. Consider the gain function G_k on every $\mathscr{D}_k$:

$\mathscr{D}_1$	$\{\omega_1\}$	$\{\omega_1,\omega_2\}$	Ω
G_1	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$
$\mathscr{D}_2$	$\{\omega_1\}$	$\{\omega_1,\omega_3\}$	Ω
G_2	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$
$\mathscr{D}_3$	$\{\omega_2\}$	$\{\omega_1,\omega_2\}$	Ω
G_3	$-\frac{1}{5}\lambda_{1,2}+\frac{4}{5}\lambda_3$	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$
$\mathscr{D}_4$	$\{\omega_2\}$	$\{\omega_2,\omega_3\}$	Ω
G_4	$-\frac{1}{5}\lambda_{1,2}+\frac{4}{5}\lambda_3$	$-\frac{1}{5}\lambda_{1,2}+\frac{4}{5}\lambda_3$	$\frac{4}{5}\lambda_{1,2,3}$
$\mathscr{D}_5$	$\{\omega_3\}$	$\{\omega_1,\omega_3\}$	Ω
G_5	$-\frac{1}{5}\lambda_{1,3}$	$\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$
$\mathscr{D}_6$	$\{\omega_3\}$	$\{\omega_2,\omega_3\}$	Ω
G_6	$-\frac{1}{5}\lambda_{1,3}$	$-\frac{1}{5}\lambda_{1,2}+\frac{4}{5}\lambda_{1,2,3}$	$\frac{4}{5}\lambda_{1,2,3}$

Choosing $\lambda_1,\lambda_2,\lambda_3$ such that $\lambda_{1,2,3} < 0$, we have that $\max_{D\in\mathscr{D}_k} G_k(D) < 0$ for $k = 1,2$. Choosing $\lambda_1,\lambda_2,\lambda_3$ such that $\lambda_{1,2,3} < 0$ and $\lambda_3 < \frac{1}{4}\lambda_{1,2}$, we have that $\max_{D\in\mathscr{D}_k} G_k(D) < 0$ for $k = 3,4$. Choosing $\lambda_1,\lambda_2,\lambda_3$ such that $\lambda_{1,2,3} < 0$ and $\lambda_{1,3} > 0$, we have that $\max_{D\in\mathscr{D}_5} G_5(D) < 0$. Finally, choosing $\lambda_1,\lambda_2,\lambda_3$ such that $\lambda_{1,2,3} < 0$, $\lambda_3 < \frac{1}{4}\lambda_{1,2}$ and $\lambda_{1,3} > 0$, we have that $\max_{D\in\mathscr{D}_6} G_6(D) < 0$. Therefore the assessment Π is not b-coherent.

On the other hand, the assessment is immediately seen not to be e-coherent as it cannot exist a conditional possibility Π' extending Π since

$$\Pi(A|\Omega) \neq \Pi(A|B) \cdot \Pi(B|\Omega).$$

The next result shows that b-coherence (e-coherence) is a necessary and sufficient condition for the extendibility of an assessment Π on $\mathscr{G}$ to a full conditional possibility Π' on $\mathscr{P}(\Omega)$. As a by-product, such result assures that every b-coherent (e-coherent) unconditional possibility assessment can be extended to a full conditional possibility on $\mathscr{P}(\Omega)$.

Theorem 6 *Let $\Pi : \mathscr{G} \to [0,1]$ be a numerical conditional possibility assessment. Then Π can be extended to a full conditional possibility Π' on $\mathscr{P}(\Omega)$ if and only if Π is b-coherent (e-coherent).*

Proof The only if part is trivial since if Π can be extended to a full conditional possibility on $\mathscr{P}(\Omega)$, then it is e-coherent and, so, b-coherent. Thus, we only prove the if part. By the equivalence between b-coherence and e-coherence proved in Theorem 5, there exists a conditional possibility defined on $\mathscr{P}(\Omega) \times \mathscr{H}$ extending Π, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions. In turn, Theorem 4 in [7] assures that such conditional possibility can be extended to a full conditional possibility Π' on $\mathscr{P}(\Omega)$. ■

4. Comparative Conditional Possibility Assessments

Let $\mathscr{B} \subseteq \mathscr{P}(\Omega)$ be a non-empty set of events and $H \in \mathscr{P}(\Omega)^0$. Consider a pair of binary relations $(\precsim_H, \prec_H)$ on $\mathscr{B} \times \{H\}$ expressing a comparative conditional possibility assessment of an agent.

For every $E|H, F|H \in \mathscr{B} \times \{H\}$, $E|H \precsim_H F|H$ stands for "E is no more possible than F under the hypothesis that H occurs", and $E|H \prec_H F|H$ stands for "F is more possible than E under the hypothesis that H occurs", respectively. As usual, $E|H \sim_H F|H$ denotes the equipossibility judgment $E|H \precsim_H F|H$ and $F|H \precsim_H E|H$. It is natural to have $\prec_H \subseteq \precsim_H \setminus \sim_H$ since, at an initial stage of judgment, the agent may not have decided yet if $E|H \prec_H F|H$ or $E|H \sim_H F|H$ and so he/she sets $E|H \precsim_H F|H$.

The relation $\precsim_H$ is *complete* if, for every $E|H, F|H \in \mathscr{B} \times \{H\}$, either $E|H \precsim_H F|H$ or $F|H \precsim_H E|H$, and *transitive* if, for every $E|H, F|H, G|H \in \mathscr{B} \times \{H\}$, $E|H \precsim_H F|H$ and $F|H \precsim_H G|H$ implies $E|H \precsim_H G|H$. Finally, $\precsim_H$ is a *weak order* if it is complete and transitive.

Obviously, if $\precsim_H$ is complete then we assume, as usual, that $\prec_H$ coincides with the asymmetric part of $\precsim_H$, i.e., $E|H \prec_H F|H$ if and only if $E|H \precsim_H F|H$ and $\neg(F|H \precsim_H E|H)$, so, we do not need to refer to the pair $(\precsim_H, \prec_H)$, as we can work with $\precsim_H$ alone.

In the following no property (such as completeness or transitivity) is required for $\precsim_H$ or $\prec_H$.

Definition 7 *A pair $(\precsim_H, \prec_H)$ of relations on a set $\mathscr{B} \times \{H\}$ of conditional events is called* **strengthened (conditional) relation** *if, for every $E|H, F|H \in \mathscr{B} \times \{H\}$, $E|H \prec_H F|H$ implies $E|H \precsim_H F|H$ and $\neg(F|H \precsim_H E|H)$, and the relation $\prec_H$ is not empty.*

Note that, the existence of at least a strict comparison is easily met, possibly by adding the non-triviality condition $\emptyset|H \prec_H \Omega|H$.

In what follows we consider an arbitrary non-empty set of conditional events $\mathscr{G} \subseteq \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)^0$ and denote again by $\mathscr{E}(\mathscr{G}) = \{H \in \mathscr{P}(\Omega)^0 : E|H \in \mathscr{G}\}$ the corresponding set of conditioning events related to $\mathscr{G}$.

We assume that the agent is able to provide a family of strengthened relations

$$\{(\precsim_H, \prec_H) : H \in \mathscr{E}(\mathscr{G})\}, \tag{10}$$

each one defined on $\mathscr{G} \cap (\mathscr{P}(\Omega) \times \{H\})$. Denote by $(\precsim, \prec)$ the pair of relations on $\mathscr{G}$ defined as

$$\precsim \;=\; \bigcup\{\precsim_H : H \in \mathscr{E}(\mathscr{G})\}, \tag{11}$$

$$\prec \;=\; \bigcup\{\prec_H : H \in \mathscr{E}(\mathscr{G})\}, \tag{12}$$

that we continue to call *strengthened relation*, since $\prec \subseteq \precsim \setminus \sim$ and $\prec$ is not empty, where $\sim$ is the symmetric relation induced by $\precsim$.

Notice that a strengthened relation $(\precsim, \prec)$ on $\mathscr{G}$ compares only events under the same conditioning event. In what follows we refer to a strengthened relation $(\precsim, \prec)$ on $\mathscr{G}$ as a comparative conditional possibility assessment.

The following definition introduces a notion of coherence in terms of representability of all weak and strict comparisons in $(\precsim, \prec)$. Notice that this is a comparative version of Definition 3 in which we require that $(\precsim, \prec)$ can be extended by a *comparative conditional possibility* $\precsim'$ induced by a conditional possibility $\Pi : \mathscr{P}(\Omega) \times \mathscr{H} \to [0,1]$. We stress that the relation $\precsim'$ is actually a weak order on $\mathscr{P}(\Omega) \times \mathscr{H}$ defined setting, for every $E|H, F|K \in \mathscr{P}(\Omega) \times \mathscr{H}$,

$$E|H \precsim' F|K \Longleftrightarrow \Pi(E|H) \leq \Pi(F|K), \tag{13}$$

thus $\precsim'$ compares also conditional events under different conditioning events. In analogy with the numerical case, we say that $\precsim'$ is *full on* $\mathscr{P}(\Omega)$ if it is induced by a full conditional possibility Π on $\mathscr{P}(\Omega)$.

Definition 8 *A comparative assessment $(\precsim, \prec)$ on $\mathscr{G}$ is said to be* **ec-coherent** *if there exists a conditional possibility $\Pi : \mathscr{P}(\Omega) \times \mathscr{H} \to [0,1]$, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions, such that, for every $E|H, F|H \in \mathscr{G}$,*

$$\begin{cases} E|H \precsim F|H \Longrightarrow \Pi(E|H) \leq \Pi(F|H), \\ E|H \prec F|H \Longrightarrow \Pi(E|H) < \Pi(F|H). \end{cases} \tag{14}$$

The following definition introduces a coherence notion analogous to that in Definition 4.

Definition 9 *A comparative assessment $(\precsim, \prec)$ on $\mathscr{G}$ is said to be* **bc-coherent** *if for every strict comparison $E|H \prec F|H$ there exists a fixed real number $\delta_{(E|H,F|H)} > 0$ such that, for every finite number of comparisons $E_1|H_1 \precsim F_1|H_1, \ldots, E_n|H_n \precsim F_n|H_n$, there exists $\mathscr{D} \in$* **chains**$(\mathscr{U}, H_0)$

with $H_0 = \bigcup_{i=1}^n H_i$, assuring that for every $\lambda_1,\ldots,\lambda_n \geq 0$, and setting $\delta_i = \delta_{(E_i|H_i,F_i|H_i)}$ if $E_i|H_i \prec F_i|H_i$ and 0 otherwise, the function $G : \mathscr{D} \to \mathbb{R}$ defined, for every $D \in \mathscr{D}$, as

$$G(D) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{F_i\cap H_i}(D) - \mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D) - \delta_i \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D)\right] \tag{15}$$

satisfies $\max_{D\in\mathscr{D}} G(D) \geq 0$.

The above condition requires the existence of a positive real number $\delta_{(E|H,F|H)}$ for every strict comparison $E|H \prec F|H$, to be interpreted as a "penalty fee" for betting on it. Also in this case, the function G introduced in Definition 9 can be regarded as the gain in a combination of conditional bets assuming partially resolving uncertainty and consonance, plus a systematically optimistic behavior. We have that the single bet on $E_i|H_i \precsim F_i|H_i$ has non-negative stake λ_i and assumes to bet in favor of the more possible conditional event and against the less possible conditional event. Further, the agent accepts to pay a "penalty fee" if the comparison is strict. The bet is called off if no information on the occurrence of H_i is obtained in the considered chain. The bc-coherence condition requires that there exists a chain $\mathscr{D}$ such that for every combination of bets on the considered comparisons the gain is not uniformly negative over $\mathscr{D}$ (no sure loss).

We stress that, Definition 9 provides a betting scheme interpretation even if the condition is not purely qualitative. A purely qualitative characterization of a coherent conditional comparative possibility relation has been given in [5] in the case $T = \min$.

Theorem 10 *For a comparative conditional possibility assessment $(\precsim, \prec)$ on $\mathscr{G}$, the following statements are equivalent:*

(i) $(\precsim, \prec)$ is ec-coherent;

(ii) $(\precsim, \prec)$ is bc-coherent.

Proof *(i) $\Longrightarrow$ (ii).* If $(\precsim, \prec)$ is ec-coherent, then there is a conditional possibility on $\Pi : \mathscr{P}(\Omega) \times \mathscr{H} \to [0,1]$ representing all the weak and strict comparisons in $(\precsim, \prec)$, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions.

For every strict comparison $E|H \prec F|H$, let $\delta_{(E|H,F|H)} = \Pi(F|H) - \Pi(E|H)$. For every finite number of comparisons $E_1|H_1 \precsim F_1|H_1, \ldots, E_n|H_n \precsim F_n|H_n$, let $H_0 = \bigcup_{i=1}^n H_i$ and set $\delta_i = \delta_{(E_i|H_i,F_i|H_i)}$ if $E_i|H_i \prec F_i|H_i$ and 0 otherwise. Then $\Pi(\cdot|H_0)$ is a possibility measure on $\mathscr{P}(\Omega)$ whose dual necessity measure has Möbius inverse m. Since $\Pi(H_0|H_0) = 1$ and $\Pi(H_0^c|H_0) = 0$, there exists a $\mathscr{D} = \{D_1,\ldots,D_h\} \in \mathbf{chains}(\mathscr{U}, H_0)$ containing the focal elements of m. Consider the matrices $\mathbf{A} = [a_{1j}] \in \mathbb{R}^{1\times h}$ and $\mathbf{B} = [b_{ij}] \in \mathbb{R}^{n\times h}$ with

$$a_{1j} = \mathbf{1}^{\mathbf{U}}_{H_0}(D_j) = 1,$$

$$b_{ij} = \mathbf{1}^{\mathbf{U}}_{F_i\cap H_i}(D_j) - \mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D_j) - \delta_i \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D_j).$$

Setting $\mathbf{x} = [m(D_1),\ldots,m(D_h)]^T \in \mathbb{R}^{h\times 1}$, we have that $\mathbf{x}$ is a solution of the following system

$$\mathscr{S} : \begin{cases} \mathbf{Ax} > 0, \\ \mathbf{Bx} \geq \mathbf{0}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases}$$

By the Motzkin's theorem of the alternative [26], the above system $\mathscr{S}$ has solution if and only if the following system has no solution

$$\mathscr{S}^* : \begin{cases} \mathbf{yA} + \mathbf{zB} \leq \mathbf{0}, \\ \mathbf{y}, \mathbf{z} \geq \mathbf{0}, \\ \mathbf{y} \neq \mathbf{0}, \end{cases}$$

where $\mathbf{y} = [y_1] \in \mathbb{R}^{1\times 1}$ and $\mathbf{z} = [\lambda_1,\ldots,\lambda_n] \in \mathbb{R}^{1\times n}$. Therefore, for every $y_1 > 0$ and $\lambda_1,\ldots,\lambda_n \geq 0$ there must exist at least an index $j \in \{1,\ldots,h\}$ such that $(\mathbf{yA}+\mathbf{zB})_j > 0$. Moreover, since $(\mathbf{yA})_j = y_1$ for every $j \in \{1,\ldots,h\}$, then the non-solvability of $\mathscr{S}^*$ is equivalent, for every $\lambda_1,\ldots,\lambda_n \geq 0$, to the existence of $j \in \{1,\ldots,h\}$ such that $(\mathbf{zB})_j \geq 0$, which, in turn, is equivalent to

$$\max_{j\in\{1,\ldots,h\}} G(D_j) \geq 0,$$

where we set

$$G(D_j) = \sum_{i=1}^{n} \lambda_i \cdot \left[\mathbf{1}^{\mathbf{U}}_{F_i\cap H_i}(D_j) - \mathbf{1}^{\mathbf{U}}_{E_i\cap H_i}(D_j) - \delta_i \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D_j)\right].$$

(ii) $\Longrightarrow$ (i). Suppose $(\precsim, \prec)$ is bc-coherent. Let $\mathscr{H}$ be the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions, whose top element is $H_0^0 = \bigcup_{H\in\mathscr{E}(\mathscr{G})} H$. We show that bc-coherence implies the existence of a $\mathscr{H}$-minimal agreeing class $\{\Pi_0,\ldots,\Pi_k\}$ on $\mathscr{P}(\Omega)$ corresponding to a conditional possibility $\Pi(\cdot|\cdot)$ on $\mathscr{P}(\Omega) \times \mathscr{H}$ representing all weak and strict comparisons in $(\precsim, \prec)$.

Let $\mathscr{R}_0 = \{(E|H, F|H) \in \mathscr{G}^2 : E|H \precsim F|H\}$ and set $n_0 = \mathrm{card}\mathscr{R}_0$. Consider an enumeration of all comparisons $E_1|H_1 \precsim F_1|H_1, \ldots, E_{n_0}|H_{n_0} \precsim F_{n_0}|H_{n_0}$ and set $\delta_i = \delta_{(E_i|H_i,F_i|H_i)}$ if $E_i|H_i \prec F_i|H_i$ and 0 otherwise. Proceeding as in the last part of the proof of the converse implication, we have that bc-coherence implies the existence of $\mathscr{D}_0 = \{D_1,\ldots,D_{h_0}\} \in \mathbf{chains}(\mathscr{U}, H_0^0)$ such that the following system has no solution

$$\mathscr{S}_0^* : \begin{cases} \mathbf{y}_0\mathbf{A}_0 + \mathbf{z}_0\mathbf{B}_0 \leq \mathbf{0}, \\ \mathbf{y}_0, \mathbf{z}_0 \geq \mathbf{0}, \\ \mathbf{y}_0 \neq \mathbf{0}, \end{cases}$$

where $\mathbf{y}_0 = [y_1] \in \mathbb{R}^{1\times 1}$, $\mathbf{z}_0 = [\lambda_1,\ldots,\lambda_{n_0}] \in \mathbb{R}^{1\times n_0}$, $\mathbf{A}_0 = [a_{1j}] \in \mathbb{R}^{1\times h_0}$ and $\mathbf{B}_0 = [b_{ij}] \in \mathbb{R}^{n_0\times h_0}$ with

$$a_{1j} = \mathbf{1}^{\mathbf{U}}_{H_0^0}(D_j) = 1,$$

$$b_{ij} \quad = \quad \mathbf{1}^{\mathbf{U}}_{F_i \cap H_i}(D_j) - \mathbf{1}^{\mathbf{U}}_{E_i \cap H_i}(D_j) - \delta_i \cdot \mathbf{1}^{\mathbf{U}}_{H_i}(D_j).$$

In turn, by the Motzkin's theorem of the alternative, the non-solvability of $\mathscr{S}^*_0$ is equivalent to the solvability of the following system

$$\mathscr{S}_0 : \begin{cases} \mathbf{A}_0\mathbf{x}_0 > 0, \\ \mathbf{B}_0\mathbf{x}_0 \geq \mathbf{0}, \\ \mathbf{x}_0 \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x}_0 = [x_1, \ldots, x_{h_0}]^T \in \mathbb{R}^{h_0 \times 1}$. Defining $m_0 : \mathscr{P}(\Omega) \to [0,1]$ by setting $m_0(D_j) = \frac{x_j}{\sum_{r=1}^{h_0} x_r}$, for $j = 1, \ldots, h_0$, and 0 otherwise, we get the Möbius inverse of a necessity measure whose dual is a possibility measure Π_0 such that $\Pi_0(H^0_0) = 1$, $\Pi_0((H^0_0)^c) = 0$, and

$$\Pi_0(F_i \cap H_i) - \Pi_0(E_i \cap H_i) - \delta_i \cdot \Pi_0(H_i) \geq 0.$$

Thus, if $\Pi_0(H_i) > 0$ we have that

$$\frac{\Pi_0(F_i \cap H_i)}{\Pi_0(H_i)} - \frac{\Pi_0(E_i \cap H_i)}{\Pi_0(H_i)} \geq \delta_i.$$

For $\alpha > 0$, let $I_\alpha = \{i \in \{1, \ldots, n_0\} : \Pi_\beta(H_i) = 0, \beta = 0, \ldots, \alpha - 1\}$. If $I_\alpha = \emptyset$ the construction stops, otherwise let $H^\alpha_0 = \bigcup_{i \in I_\alpha} H_i$ and set $\mathscr{R}_\alpha = \{(E_i|H_i, F_i|H_i)\}_{i \in I_\alpha}$ where $n_\alpha = \mathrm{card}\mathscr{R}_\alpha$. Fix the enumeration $E_{k_1}|H_{k_1} \precsim F_{k_1}|H_{k_1}, \ldots, E_{k_{n_\alpha}}|H_{k_{n_\alpha}} \precsim F_{k_{n_\alpha}}|H_{k_{n_\alpha}}$ and set $\delta_i = \delta_{(E_{k_i}|H_{k_i}, F_{k_i}|H_{k_i})}$ if $E_{k_i}|H_{k_i} \prec F_{k_i}|H_{k_i}$ and 0 otherwise. Again, we have that bc-coherence implies the existence of $\mathscr{D}_\alpha = \{D_1, \ldots, D_{h_\alpha}\} \in \mathbf{chains}(\mathscr{U}, H^\alpha_0)$ such that the following system has no solution

$$\mathscr{S}^*_\alpha : \begin{cases} \mathbf{y}_\alpha\mathbf{A}_\alpha + \mathbf{z}_\alpha\mathbf{B}_\alpha \leq \mathbf{0}, \\ \mathbf{y}_\alpha, \mathbf{z}_\alpha \geq \mathbf{0}, \\ \mathbf{y}_\alpha \neq \mathbf{0}, \end{cases}$$

where $\mathbf{y}_\alpha = [y_1] \in \mathbb{R}^{1\times 1}$, $\mathbf{z}_\alpha = [\lambda_1, \ldots, \lambda_{n_\alpha}] \in \mathbb{R}^{1 \times n_\alpha}$, $\mathbf{A}_\alpha = [a_{1j}] \in \mathbb{R}^{1 \times h_\alpha}$ and $\mathbf{B}_\alpha = [b_{ij}] \in \mathbb{R}^{n_\alpha \times h_\alpha}$ with

$$\begin{aligned} a_{1j} &= \mathbf{1}^{\mathbf{U}}_{H^\alpha_0}(D_j) = 1, \\ b_{ij} &= \mathbf{1}^{\mathbf{U}}_{F_{k_i} \cap H_{k_i}}(D_j) - \mathbf{1}^{\mathbf{U}}_{E_{k_i} \cap H_{k_i}}(D_j) - \delta_i \cdot \mathbf{1}^{\mathbf{U}}_{H_{k_i}}(D_j). \end{aligned}$$

In turn, by the Motzkin's theorem of the alternative, the non-solvability of $\mathscr{S}^*_\alpha$ is equivalent to the solvability of the following system

$$\mathscr{S}_\alpha : \begin{cases} \mathbf{A}_\alpha\mathbf{x}_\alpha > 0, \\ \mathbf{B}_\alpha\mathbf{x}_\alpha \geq \mathbf{0}, \\ \mathbf{x}_\alpha \geq \mathbf{0}, \end{cases}$$

where $\mathbf{x}_\alpha = [x_1, \ldots, x_{h_\alpha}]^T \in \mathbb{R}^{h_\alpha \times 1}$. Defining $m_\alpha : \mathscr{P}(\Omega) \to [0,1]$ by setting $m_\alpha(D_j) = \frac{x_j}{\sum_{r=1}^{h_\alpha} x_r}$, for $j = 1, \ldots, h_\alpha$, and 0 otherwise, we get the Möbius inverse of a necessity measure whose dual is a possibility measure Π_α such that $\Pi_\alpha(H^\alpha_0) = 1$, $\Pi_\alpha((H^\alpha_0)^c) = 0$, and

$$\Pi_\alpha(F_{k_i} \cap H_{k_i}) - \Pi_\alpha(E_{k_i} \cap H_{k_i}) - \delta_i \cdot \Pi_\alpha(H_{k_i}) \geq 0.$$

Thus, if $\Pi_\alpha(H_{k_i}) > 0$ we have that

$$\frac{\Pi_\alpha(F_{k_i} \cap H_{k_i})}{\Pi_\alpha(H_{k_i})} - \frac{\Pi_\alpha(E_{k_i} \cap H_{k_i})}{\Pi_\alpha(H_{k_i})} \geq \delta_i.$$

Let k be the first index such that $I_{k+1} = \emptyset$. Then $\{\Pi_0, \ldots, \Pi_k\}$ is by construction a $\mathscr{H}$-minimal agreeing class corresponding to a conditional possibility $\Pi(\cdot|\cdot)$ on $\mathscr{P}(\Omega) \times \mathscr{H}$ representing all weak and strict comparisons in $(\precsim, \prec)$. ■

Also in the comparative case, by identifying every unconditional event E with the conditional event $E|\Omega$, the bc-coherence condition for an unconditional assessment $(\precsim, \prec)$ on $\mathscr{G}$ with $\mathscr{G} \subseteq \mathscr{P}(\Omega)$ reduces to: for every strict comparison $E \prec F$ there exists a fixed real number $\delta_{(E,F)} > 0$ such that, for every finite number of comparisons $E_1 \precsim F_1, \ldots, E_n \precsim F_n$, there exists $\mathscr{D} \in \mathbf{chains}(\mathscr{U}, \Omega)$, assuring that for every $\lambda_1, \ldots, \lambda_n \geq 0$, and setting $\delta_i = \delta_{(E_i,F_i)}$ if $E_i \prec F_i$ and 0 otherwise, the function $G : \mathscr{D} \to \mathbb{R}$ defined, for every $D \in \mathscr{D}$, as

$$G(D) = \sum_{i=1}^{n} \lambda_i \cdot [\mathbf{1}^{\mathbf{U}}_{F_i}(D) - \mathbf{1}^{\mathbf{U}}_{E_i}(D) - \delta_i] \qquad (16)$$

satisfies $\max_{D \in \mathscr{D}} G(D) \geq 0$.

Notice that, a weak order $\precsim$ on an algebra of events $\mathscr{G} \subseteq \mathscr{P}(\Omega)$ is bc-coherent if and only if it is a comparative possibility relation in the sense of Dubois [15].

Example 2 *Let Ω, A and B as in Example 1 and consider an assessment $(\precsim, \prec)$ such that $B|\Omega \precsim A|\Omega$, $\emptyset|\Omega \prec A|\Omega$, $A|B \prec \Omega|B$. We refer to a combination of bets involving the three given comparisons. Let $H_0 = \Omega$ and consider* $\mathbf{chains}(\mathscr{U}, H_0) = \{\mathscr{D}_1, \ldots, \mathscr{D}_6\}$ *as in Example 1. For this assessment to be bc-coherent there must exist fixed positive numbers $\delta_{(\emptyset|\Omega, A|\Omega)}$ and $\delta_{(A|B, \Omega|B)}$ to be interpreted as "penalty fees" for betting on strict comparisons. Let $\delta_1 = 0$, $\delta_2 = \delta_{(\emptyset|\Omega,A|\Omega)}$ and $\delta_3 = \delta_{(A|B,\Omega|B)}$, and denote $\overline{\delta_i} = 1 - \delta_i$.*

Consider the gain function G_k on every $\mathscr{D}_k$:

$\mathscr{D}_1$	$\{\omega_1\}$	$\{\omega_1, \omega_2\}$	Ω
G_1	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$
$\mathscr{D}_2$	$\{\omega_1\}$	$\{\omega_1, \omega_3\}$	Ω
G_2	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$
$\mathscr{D}_3$	$\{\omega_2\}$	$\{\omega_1, \omega_2\}$	Ω
G_3	$-\lambda_1 - \lambda_2\delta_2 + \lambda_3\overline{\delta_3}$	$\lambda_2\overline{\delta_2} + \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$
$\mathscr{D}_4$	$\{\omega_2\}$	$\{\omega_2, \omega_3\}$	Ω
G_4	$-\lambda_1 - \lambda_2\delta_2 + \lambda_3\overline{\delta_3}$	$-\lambda_1 - \lambda_2\delta_2 + \lambda_3\overline{\delta_3}$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$
$\mathscr{D}_5$	$\{\omega_3\}$	$\{\omega_1, \omega_3\}$	Ω
G_5	$-\lambda_2\delta_2 - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$
$\mathscr{D}_6$	$\{\omega_3\}$	$\{\omega_2, \omega_3\}$	Ω
G_6	$-\lambda_2\delta_2$	$-\lambda_1 - \lambda_2\delta_2 + \lambda_3\overline{\delta_3}$	$\lambda_2\overline{\delta_2} - \lambda_3\delta_3$

Choosing $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\lambda_3 > \max\left\{\frac{\overline{\delta_2}}{\delta_3}\lambda_2, 0\right\}$, we have that $\max_{D \in \mathscr{D}_k} G_k(D) < 0$ for $k = 1, 2, 5$. Choosing

$\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\lambda_3 > \max\left\{\frac{\overline{\delta_2}}{\delta_3}\lambda_2, 0\right\}$ and $\lambda_1 > \max\{-\lambda_2\delta_2 + \lambda_3\overline{\delta_3}, 0\}$, we have that $\max_{D \in \mathscr{D}_k} G_k(D) < 0$ for $k = 3,4$. Choosing $\lambda_1, \lambda_2, \lambda_3 \geq 0$ such that $\lambda_2 > 0$, $\lambda_3 > \max\left\{\frac{\overline{\delta_2}}{\delta_3}\lambda_2, 0\right\}$ and $\lambda_1 > \max\{-\lambda_2\delta_2 + \lambda_3\overline{\delta_3}, 0\}$, we have that $\max_{D \in \mathscr{D}_6} G_6(D) < 0$. This implies that there cannot exist positive $\delta_{(\emptyset|\Omega, A|\Omega)}$ and $\delta_{(A|B,\Omega|B)}$ satisfying Definition 9. Therefore the assessment $(\precsim, \prec)$ is not bc-coherent.

On the other hand, the assessment is immediately seen not to be ec-coherent since every conditional possibility Π representing the given comparisons is such that $\Pi(A|\Omega) = \Pi(B|\Omega) > 0$ and $\Pi(A|B) < 1$, therefore

$$\Pi(A|\Omega) \neq \Pi(A|B) \cdot \Pi(B|\Omega).$$

The next result shows that bc-coherence (ec-coherence) is a necessary and sufficient condition for the extendibility of an assessment $(\precsim, \prec)$ on $\mathscr{G}$ to a full comparative conditional possibility $\precsim'$ on $\mathscr{P}(\Omega)$ induced by a full conditional possibility Π on $\mathscr{P}(\Omega)$. As a by-product, such result assures that every bc-coherent (ec-coherent) comparative unconditional possibility assessment can be extended to a full comparative conditional possibility on $\mathscr{P}(\Omega)$.

Theorem 11 *Let $(\precsim, \prec)$ on $\mathscr{G}$ be a comparative conditional possibility assessment. Then $(\precsim, \prec)$ can be extended to a full comparative conditional possibility $\precsim'$ on $\mathscr{P}(\Omega)$ if and only if $(\precsim, \prec)$ is bc-coherent (ec-coherent).*

Proof The only if part is trivial since if $(\precsim, \prec)$ can be extended to a full comparative conditional possibility $\precsim'$ on $\mathscr{P}(\Omega)$, then it is ec-coherent and, so, bc-coherent. Thus, we only prove the if part. By the equivalence between bc-coherence and ec-coherence proved in Theorem 10, there exists a conditional possibility defined on $\mathscr{P}(\Omega) \times \mathscr{H}$ representing all the weak an strict comparisons in $(\precsim, \prec)$, where $\mathscr{H}$ is the additive class obtained closing $\mathscr{E}(\mathscr{G})$ with respect to unions. In turn, Theorem 4 in [7] assures that such conditional possibility can be extended to a full conditional possibility on $\mathscr{P}(\Omega)$. Finally, such full conditional possibility induces the searched full comparative conditional possibility $\precsim'$ on $\mathscr{P}(\Omega)$. ■

5. Conclusions

We introduce two coherence conditions (namely, b-coherence and bc-coherence) for numerical and comparative conditional possibility assessments, proving they are equivalent to the extendibility of the given assessment to a numerical or comparative conditional possibility defined on a structured domain. Both conditions provide an operational tool to elicit a subjective numerical or comparative conditional possibility assessment and rule the extension of the given assessment to any larger domain.

The generalization of the present results to arbitrary sets of conditional events will be the aim of future research. Another possible generalization, limited to the comparative case, consists in allowing comparisons of conditional events under different conditioning events. Further, the proposed conditions can be reformulated interpreting events as purely logical entities, that is by referring to abstract Boolean algebras, as in de Finetti's approach to probability.

We point out that, if in Definitions 4 and 9 we define the gain G on $\mathscr{U}_0 = \{B \in \mathscr{U} : B \subseteq H_0\}$ and require that $\max_{B \in \mathscr{U}_0} G(B) \geq 0$, then we assume that the agent bets under partially resolving uncertainty but not under consonance. In this case, it is possible to show (see [10]) that the resulting coherence conditions refer to the axiomatic definition of conditional plausibility function given in [6]. In analogy, if in Definitions 4 and 9 we define the gain G on $\mathscr{C}_0 = \{\{\omega\} \in \mathscr{U} : \omega \in H_0\}$ and require that $\max_{\{\omega\} \in \mathscr{C}_0} G(\{\omega\}) \geq 0$, then we assume that the agent bets under completely resolving uncertainty. In this case, the resulting coherence conditions refer to the axiomatic definition of conditional probability due to Dubins [14].

Acknowledgments

The authors are members of the GNAMPA-INdAM research group. The first author was supported by Università degli Studi di Perugia, Fondo Ricerca di Base 2019, project "Modelli per le decisioni economiche e finanziarie in condizioni di ambiguità ed imprecisione".

References

[1] B. Bouchon-Meunier, G. Coletti, and C. Marsala. Independence and Possibilistic Conditioning. *Annals of Mathematics and Artificial Intelligence*, 35(1):107–123, 2002.

[2] G. Coletti and D. Petturiti. Finitely maxitive T-conditional possibility theory: Coherence and extension. *International Journal of Approximate Reasoning*, 71:64–88, 2016.

[3] G. Coletti and R. Scozzafava. *Probabilistic Logic in a Coherent Setting*, volume 15 of *Trends in Logic*. Kluwer Academic Publisher, Dordrecht/Boston/London, 2002.

[4] G. Coletti and B. Vantaggi. Possibility theory: conditional independence. *Fuzzy Sets and Systems*, 157(3): 1491–1513, 2006.

[5] G. Coletti and B. Vantaggi. Comparative models ruled by possibility and necessity: A conditional world. *International Journal of Approximate Reasoning*, 45(2): 341–363, 2007.

[6] G. Coletti and B. Vantaggi. A view on conditional measures through local representability of binary relations. *International Journal of Approximate Reasoning*, 47(1):268–283, 2008.

[7] G. Coletti and B. Vantaggi. T-conditional possibilities: Coherence and inference. *Fuzzy Sets and Systems*, 160(3):306–324, 2009.

[8] G. Coletti, R. Scozzafava, and B. Vantaggi. Inferential processes leading to possibility and necessity. *Informtion Sciences*, 245(1):132–145, 2013.

[9] G. Coletti, D. Petturiti, and B. Vantaggi. When upper conditional probabilities are conditional possibility measures. *Fuzzy Sets and Systems*, 304:45–64, 2016.

[10] G. Coletti, D. Petturiti, and B. Vantaggi. A Dutch book coherence condition for conditional completely alternating Choquet expectations. *Bollettino dell'Unione Matematica Italiana*, 13(4):585–593, 2020.

[11] G. de Cooman. Possibility theory II: Conditional possibility. *International Journal of General Systems*, 25(4):325–351, 1997.

[12] G. de Cooman and D. Aeyels. Supremum preserving upper probabilities. *Information Sciences*, 118(1): 173–212, 1999.

[13] A.P. Dempster. Upper and Lower Probabilities Induced by a Multivalued Mapping. *Annals of Mathematical Statistics*, 38(2):325–339, 1967.

[14] L.E. Dubins. Finitely additive conditional probabilities, conglomerability and disintegrations. *The Annals of Probability*, 3(1):89–99, 1975.

[15] D. Dubois. Belief structures, possibility theory and decomposable confidence measures on finite sets. *Computer and Artificial Intelligence*, 5:403–416, 1986.

[16] D. Dubois and H. Prade. *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum Press, New York and London, 1988.

[17] D. Dubois and H. Prade. When upper probabilities are possibility measures. *Fuzzy Sets and Systems*, 49 (1):65–74, 1992.

[18] D. Dubois and H. Prade. Bayesian conditioning in possibility theory. *Fuzzy Sets and Systems*, 92(2): 223–240, 1997.

[19] D. Dubois, S. Moral, and H. Prade. A Semantics for Possibility Theory Based on Likelihoods. *Journal of Mathematical Analysis and Applications*, 205(2): 359–380, 1997.

[20] D. Dubois, H. Prade, and P. Smets. A definition of subjective possibility. *International Journal of Approximate Reasoning*, 48(2):352–364, 2008.

[21] L. Ferracuti and B. Vantaggi. Independence and conditional possibility for strictly monotone triangular norms. *International Journal of Intelligent Systems*, 21(3):299–323, 2006.

[22] I. Gilboa and D. Schmeidler. Additive representations of non-additive measures and the Choquet integral. *Annals of Operations Research*, 52(1):43–65, 1994.

[23] R. Giles. Foundations for a theory of possibility. In M.M. Gupta and E. Sanchez, editors, *Fuzzy Information and Decision Processes*, pages 183–195. North-Holland, 1982.

[24] M. Grabisch. *Set Functions, Games and Capacities in Decision Making*. Springer, 2016.

[25] J.-Y. Jaffray. Coherent bets under partially resolving uncertainty and belief functions. *Theory and Decision*, 26(2):99–105, 1989.

[26] O.L. Mangasarian. *Nonlinear Programming*, volume 10 of *Classics in Applied Mathematics*. SIAM, 1994.

[27] E. Miranda, I. Couso, and P. Gil. A random set characterization of possibility measures. *Information Sciences*, 168(1-4):51–75, 2004.

[28] H.T. Nguyen and B. Bouchon-Meunier. Random sets and large deviations principle as a foundation for possibility measures. *Soft Computing*, 8:61–70, 2003.

[29] D. Petturiti and B. Vantaggi. Conditional submodular Choquet expected values and conditional coherent risk measures. *International Journal of Approximate Reasoning*, 113:14–38, 2019.

[30] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, Princeton, NJ, 1976.

[31] P. Smets. The degree of belief in a fuzzy event. *Information Sciences*, 25(1):1–19, 1981.

[32] P. Walley. Coherent lower (and upper) probabilities. Technical report, Department of Statistics, University of Warwick, 1981.

[33] P. Walley and G. de Cooman. Coherence of rules for defining conditional possibility. *International Journal of Approximate Reasoning*, 21(1):63–107, 1999.

[34] L.A Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1(1):3–28, 1978.