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A Remarkable Equivalence between Non-Stationary Precise and Stationary Imprecise Uncertainty Models in Computable Randomness

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Abstract

The field of algorithmic randomness studies what it means for infinite binary sequences to be random for some given uncertainty model. Classically, such randomness involves precise uncertainty models, and it is only recently that imprecision has been introduced into this field. As a consequence, the investigation into how imprecision alters our view on random sequences has only just begun. In this contribution, we establish a close and surprising connection between precise and imprecise uncertainty models in this randomness context. In particular, we show that there are stationary imprecise models and non-stationary precise models that have the exact same set of computably random sequences. We also discuss the possible implications of this result for a statistics based on imprecise probabilities.

Keywords: computable randomness, imprecise probabilities, computability, coherent upper expectations, supermartingales, non-stationarity

1. Introduction

What does it mean for an infinite binary sequence 011001100... to be random? This is a highly non-trivial question that has led to numerous investigations. First of all, it is important to realise that randomness is typically defined with respect to some uncertainty model. So, our opening question only makes sense once such a model has been specified. These uncertainty models can be stationary or non-stationary, as well as precise or imprecise [4, 5, 6, 11]. It is between the non-stationary precise and the stationary imprecise uncertainty models that we will reveal a remarkably close connection: we will show that there are stationary imprecise models and non-stationary precise models that have the exact same set of computably random sequences.

Historically, the earliest notion of randomness—called Church randomness—only considered precise probability models that assign a probability $p \in [0,1]$ to the outcome 1. According to this notion, an infinite binary sequence is Church random for p if the relative frequency of ones along every computably selectable infinite subsequence converges to p [1, 3], where 'computably selectable' essentially means that there is some finite algorithm that decides which elements to keep and to discard.

However, there are infinite sequences that satisfy this requirement, but for which the running frequency of ones along the sequence converges to p from below. Obviously, such sequences disobey the law of the iterated logarithm. For this reason, Jean Ville criticised this randomness definition, and argued that besides the law of large numbers, a random sequence also ought to satisfy other statistical laws [1]. Such discussions led to the development of many other notions of randomness.

The most well-known and well-studied notions among those are Martin-Löf randomness, computable randomness and Schnorr randomness. The reason for that is twofold: they have an intuitive interpretation and they can be defined in several equivalent ways [7]. From a measure-theoretic point of view, for example, an infinite binary sequence is random for $p \in [0,1]$ if it passes all computably implementable statistical tests that are associated with p. On the other hand, if we adopt the martingale-theoretic approach, then a sequence is random for p if there is no computable betting strategy for getting arbitrarily rich along this sequence without borrowing, where the bets that are allowed are determined by p, and where computability again means that there is a finite algorithm that yields the strategy.

There is more to randomness, though, than the simple case of a single probability p. As we mentioned above, more general uncertainty models, such as non-stationary precise ones, can also be used to define notions of randomness [11]. And it is only recently that imprecise-probabilistic uncertainty models have been introduced in this context. That is, De Cooman and De Bock put forward a martingale-theoretic approach that allows us to associate computable randomness with imprecise rather than precise probability models [4, 5, 6]. Their work still leaves room for many open questions on how allowing for imprecision changes our understanding of random sequences. In this paper, we contribute to this understanding by proving a remarkable relation between randomness for precise and imprecise probability models. In particular, for every non-singular rational interval $I \subseteq [0,1]$, we will show that there is a non-stationary precise but non-computable uncertainty model for which the set of computably random paths is the same.

Our contribution is structured as follows. We start by introducing interval forecasts and coherent upper expectations in Section 2, and explain how they can be interpreted in terms of gambles that a subject is willing to offer. Section 3 explains how to bet on a single variable in a way that agrees with these uncertainty models, and lifts this idea to a betting game/protocol on an infinite sequence of variables by defining betting strategies that again agree with these uncertainty models, and avoid borrowing. After clarifying in Section 4 when such betting strategies are computable, we present in Section 5 the imprecise-probabilistic notion of computable randomness that we introduced in earlier work [4, 8] and discuss some of its properties. Finally, in Section 6, we prove our central result; an infinite sequence is computably random for a rational interval forecast if and only if it is computably random for some specific related non-computable non-stationary precise uncertainty model. In Section 7, we elaborate on the possible implications of this result for a statistics based on imprecise probabilities.

2. Interval Forecasts

We start by considering a single variable X that takes values x in the binary outcome space $\mathscr{X} := \{1,0\}$. To describe a subject's uncertainty about the unknown value of X, we use a closed interval $I \subseteq [0,1]$. We collect all such closed intervals in the set $\mathscr{I}$ and call them *interval forecasts*. One way to interpret an interval forecast $I \in \mathscr{I}$ is to regard its elements $p \in I$ as possible values for the probability that X equals 1. In this paper, however, where betting will play a central role, we prefer to adopt a different interpretation. We interpret the lower and upper bound of I as a subject's largest acceptable buying and smallest acceptable selling price, respectively,[1] for the uncertain pay-off $X \in \{0,1\}$, expressed in some linear utility scale.

Consequently, if $I = [\underline{p}, \overline{p}]$, our subject is willing to accept the uncertain pay-off $X - p$ for any buying price $p \leq \underline{p}$, and is willing to accept the uncertain pay-off $q - X$ for any selling price $q \geq \overline{p}$. Due to the linearity of our utility scale, this implies that he is willing to accept the uncertain pay-off $\alpha(X-p) + \beta(q-X)$ for any $p \leq \underline{p}$, $q \geq \overline{p}$ and $\alpha, \beta \geq 0$. From the perspective of an opponent that bets against our subject, this means that our subject is willing to offer her any uncertain reward of the form $\alpha(p-X) + \beta(X-q)$, with $p \leq \underline{p}$, $q \geq \overline{p}$ and $\alpha, \beta \geq 0$. To manipulate these uncertain rewards mathematically, it will be convenient to identify them with maps on $\mathscr{X}$, whose value in x is obtained by replacing X with x. The reward X, for example, then corresponds to the identity map on $\mathscr{X}$. We will call any such map $f \colon \mathscr{X} \to \mathbb{R}$ from the binary sample space to the real numbers a *gamble*, and we denote the set of all gambles by $\mathscr{L}(\mathscr{X})$. Since $|\mathscr{X}| = 2$, gambles can be drawn in a two-dimensional space. This allows us to visualise the cone of gambles that is offered to an opponent; we illustrate this in Figure 1.

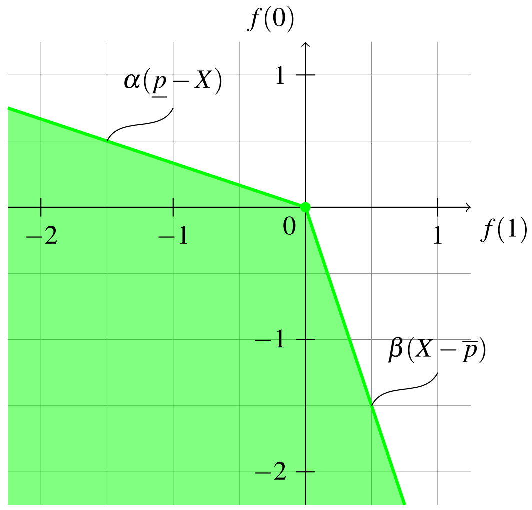

Figure 1: Let $\underline{p} := 1/4$ and $\overline{p} := 3/4$. Then the green region depicts all gambles $f \in \mathscr{L}(\mathscr{X})$ that correspond to an uncertain reward $\alpha(p-X) + \beta(X-q)$, with $p \leq \underline{p}$, $q \geq \overline{p}$ and $\alpha, \beta \geq 0$.

In what follows, it will be useful to have an analytical condition that, for a subject with interval forecast $I \in \mathscr{I}$, characterizes the gambles he is willing to offer to an opponent. To this end, we introduce upper (and lower) expectation operators. When $I = p \in \mathbb{R}$, i.e., when I reduces to a single number, we consider the linear expectation E_p defined by

$$E_p(f) := pf(1) + (1-p)f(0) \text{ for all } f \in \mathscr{L}(\mathscr{X}). \quad (1)$$

This is a most informative—or least conservative—model for a subject's uncertainty. When $I = [\underline{p}, \overline{p}] \notin \mathbb{R}$, we consider the *upper expectation* $\overline{E}_I$ defined by

$$\begin{aligned} \overline{E}_I(f) :=& \max_{p \in I} \{pf(1) + (1-p)f(0)\} \\ =& \max\{\underline{p}f(1) + (1-\underline{p})f(0), \\ & \qquad \overline{p}f(1) + (1-\overline{p})f(0)\} \\ =& \max\{E_{\underline{p}}(f), E_{\overline{p}}(f)\} \text{ for all } f \in \mathscr{L}(\mathscr{X}). \quad (2) \end{aligned}$$

1. Traditionally, in imprecise probabilities, the lower and upper bound of I are interpreted as a subject's supremum acceptable buying and infimum acceptable selling price for the uncertain pay-off X. However, as was proved in [8], our imprecise-probabilistic notion of computable randomness is the same under both interpretations. Hence, we choose the interpretation that leads to the simpler proofs.

As a closely related operator, we consider the conjugate *lower expectation* $\underline{E}_I\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ defined by

$$\underline{E}_I(f) := -\overline{E}_I(-f) = \min\{E_{\underline{p}}(f), E_{\overline{p}}(f)\} \tag{3}$$

for all $f \in \mathscr{L}(\mathscr{X})$. It is a matter of straightforward verification that the upper expectation $\overline{E}_I$ satisfies the following so-called coherence properties [2].

Proposition 1 *Consider any interval forecast $I \in \mathscr{I}$. Then for all gambles $f, g \in \mathscr{L}(\mathscr{X})$, and all $\mu \in \mathbb{R}$ and $\lambda \geq 0$:*

C1. $\min f \leq \overline{E}_I(f) \leq \max f$ [boundedness]

C2. $\overline{E}_I(\lambda f) = \lambda \overline{E}_I(f)$ [non-negative homogeneity]

C3. $\overline{E}_I(f+g) \leq \overline{E}_I(f) + \overline{E}_I(g)$ [subadditivity]

C4. $\overline{E}_I(f+\mu) = \overline{E}_I(f) + \mu$ [constant additivity]

The coherence properties C2-C4 allow us to show that a coherent upper expectation $\overline{E}_I$ indeed characterizes the gambles that are offered by our subject.

Proposition 2 *Consider any gamble $f \in \mathscr{L}(\mathscr{X})$ and any interval forecast $I = [\underline{p}, \overline{p}] \in \mathscr{I}$. Then $\overline{E}_I(f) \leq 0$ if and only if there are $p \leq \underline{p}$, $q \geq \overline{p}$ and $\alpha, \beta \geq 0$ such that $f = \alpha(p - X) + \beta(X - q)$.*

Proof of Proposition 2 To prove the direct implication, assume that $\overline{E}_I(f) \leq 0$. Observe that f can always be written as $f = \gamma(X - c)$, for suitably chosen $\gamma, c \in \mathbb{R}$. If $\gamma = 0$, then $f = 0$, so the statement then holds trivially. Consequently, it suffices to prove that if $\gamma > 0$, then $c \geq \overline{p}$, and if $\gamma < 0$, then $c \leq \underline{p}$. To do so, observe that by Equations (2) and (3) it holds that $\underline{E}_I(X) = \underline{p}$ and $\overline{E}_I(X) = \overline{p}$. Now, if $\gamma > 0$, it follows from C2 and C4 that

$$0 \geq \overline{E}_I(f) = \overline{E}_I(\gamma(X - c)) = \gamma(\overline{E}_I(X) - c) = \gamma(\overline{p} - c),$$

and hence, $c \geq \overline{p}$. If $\gamma < 0$, it follows from C2, C4 and conjugacy that

$$\begin{aligned} 0 \geq \overline{E}_I(f) &= \overline{E}_I(\gamma(X - c)) = \overline{E}_I(-\gamma(c - X)) \\ &= -\gamma \overline{E}_I(c - X) = -\gamma(c + \overline{E}_I(-X)) \\ &= -\gamma(c - \underline{E}_I(X)) = -\gamma(c - \underline{p}), \end{aligned}$$

and hence, $c \leq \overline{p}$.

To prove the converse implication, assume that $f = \alpha(p - X) + \beta(X - q)$, with $p \leq \underline{p}$, $q \geq \overline{p}$ and $\alpha, \beta \geq 0$. From C2-C4 and conjugacy it then immediately follows that

$$\begin{aligned} \overline{E}_I(f) &= \overline{E}_I(\alpha(p - X) + \beta(X - q)) \\ &\leq \overline{E}_I(\alpha(p - X)) + \overline{E}_I(\beta(X - q)) \\ &= \alpha \overline{E}_I(p - X) + \beta \overline{E}_I(X - q) \\ &= \alpha(p + \overline{E}_I(-X)) + \beta(\overline{E}_I(X) - q) \\ &= \alpha(p - \underline{E}_I(X)) + \beta(\overline{E}_I(X) - q) \\ &= \alpha(p - \underline{p}) + \beta(\overline{p} - q) \leq 0, \end{aligned}$$

which completes the proof. ■

3. Forecasting Systems and Betting Strategies

We can test the correspondence between a subject's interval forecasts and the unknown outcomes of binary variables by taking him up on a betting game.

We first introduce a betting game on a single binary variable X. There are three players involved: Forecaster, Sceptic and Reality. Forecaster initiates the game by providing an interval forecast $I \subseteq [0,1]$, which describes, as we explained in the previous section, his uncertainty about the uncertain outcome $X \in \mathscr{X}$. Next, Sceptic, being Forecaster's opponent, is allowed to pick any gamble $f \in \mathscr{L}(\mathscr{X})$ that Forecaster is willing to offer, meaning that $\overline{E}_I(f) \leq 0$. This leads to an uncertain (possibly negative) gain $f(X)$ for Sceptic and $-f(X)$ for Forecaster. Finally, Reality reveals the outcome $x \in \mathscr{X}$, which leads to an actual (possibly negative) gain $f(x)$ for Sceptic and $-f(x)$ for Forecaster.

To extend these ideas to an infinite betting game on subsequent binary variables $X_1, \dots, X_n, \dots$, we require a bit more terminology.

An infinite outcome sequence $(x_1, \dots, x_n, \dots)$ is called a *path* and is also denoted by ω. All such paths are collected in the set $\Omega := \mathscr{X}^{\mathbb{N}}$,[2] and for every path $\omega = (x_1, \dots, x_n, \dots) \in \Omega$, we let $\omega_{1:n} := (x_1, \dots, x_n)$ and $\omega_{n+1} := x_{n+1}$ for all $n \in \mathbb{N}_0$. A finite outcome sequence $x_{1:n} := (x_1, \dots, x_n) \in \mathscr{X}^n$ is called a *situation* and is also denoted by s, with length $|s| := n$. All situations are collected in the set $\mathbb{S} := \bigcup_{n \in \mathbb{N}_0} \mathscr{X}^n$. For any $s = (x_1, \dots, x_n) \in \mathbb{S}$ and $x \in \mathscr{X}$, we write sx as a shorthand notation for $(x_1, \dots, x_n, x)$. By convention, we call the empty sequence $\square := x_{1:0} = ()$ the *initial situation*. Note that for every path $\omega \in \Omega$, we have that $\omega_{1:0} = \square$.

Forecaster's part in the game consists in providing an interval forecast $I_{x_{1:n}} \in \mathscr{I}$ for every finite outcome sequence $x_{1:n} \in \mathbb{S}$, with $n \in \mathbb{N}_0$, in order to describe his uncertainty about the binary variable X_{n+1} given that he has observed $x_{1:n}$.

Definition 3 (Forecasting system) *A* forecasting system *is a map $\varphi\colon \mathbb{S} \to \mathscr{I}$ that associates with every situation $s \in \mathbb{S}$ an interval forecast $\varphi(s) \in \mathscr{I}$. A forecasting system φ is called* precise *if $\varphi(s) \in \mathbb{R}$ for all $s \in \mathbb{S}$. We denote the set $\mathscr{I}^{\mathbb{S}}$ of all forecasting systems by Φ.*

Once Forecaster has specified a forecasting system $\varphi \in \Phi$, Sceptic is allowed to adopt any betting strategy that, for every situation $s \in \mathbb{S}$, engages in a gamble $f_s \in \mathscr{L}(\mathscr{X})$ that Forecaster is bound to offer by his specification of the interval forecast $\varphi(s) \in \mathscr{I}$, i.e., a gamble f_s for which $\overline{E}_{\varphi(s)}(f_s) \leq 0$. Afterwards, Reality reveals the successive outcomes $X_n = x_n$ at each successive *time instant* $n \in \mathbb{N}$, leading up to the sequence $\omega = (x_1, \dots, x_n, \dots)$. At every

2. $\mathbb{N}$ denotes the set of natural numbers, whereas $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ denotes the set of non-negative integers.

time instant n, after Reality has revealed the finite outcome sequence $x_{1:n}$, Sceptic uses the gamble $f_{x_{1:n}}$ that corresponds to her betting strategy. Next, Reality reveals the next outcome $X_{n+1} = x_{n+1} \in \mathscr{X}$ and the reward $f_{x_{1:n}}(x_{n+1})$ goes to Sceptic. Moreover, we will prohibit Sceptic from borrowing.

To formalise these betting strategies for Sceptic, we define a *process* as a map on situations. In particular, a real process $F\colon \mathbb{S} \to \mathbb{R}$ is a map from situations to real numbers, and similarly for a rational process $F\colon \mathbb{S} \to \mathbb{Q}$. A real—or rational—process F is called non-negative if $F(s) \geq 0$ for all $s \in \mathbb{S}$; it is called positive if $F(s) > 0$ for all $s \in \mathbb{S}$. A zero-one valued process S—with $S(s) \in \{0,1\}$ for all $s \in \mathbb{S}$—is called a *selection process*.

A *gamble process* is a map from situations to gambles. In particular, we associate with every real process F a *process difference* $\Delta F\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$, which is a gamble process that maps any $s \in \mathbb{S}$ to the gamble $\Delta F(s) := F(s\bullet) - F(s)$, where we use $F(s\bullet)$ to denote the gamble on $\mathscr{X}$ whose value, for any $x \in \mathscr{X}$, is given by $F(sx)$. Note that $F(x_{1:n}) = F(\square) + \sum_{k=0}^{n-1} \Delta F(x_{1:k})(x_{k+1})$ for all $x_{1:n} \in \mathbb{S}$, with $n \in \mathbb{N}_0$. Given a forecasting system $\varphi \in \Phi$, we call a real process M a *supermartingale* for φ if $\overline{E}_{\varphi(s)}(\Delta M(s)) \leq 0$ for all $s \in \mathbb{S}$. All supermartingales for φ are collected in the set $\overline{\mathbb{M}}(\varphi)$.

Supermartingales correspond to Sceptic's allowed betting strategies. Indeed, assume that Forecaster adopts the forecasting system $\varphi \in \Phi$, consider a time instant $n \in \mathbb{N}_0$, and consider the situation where Reality has revealed a finite outcome sequence $\omega_{1:n} \in \mathbb{S}$. A supermartingale M for φ then specifies a gamble $\Delta M(\omega_{1:n}) \in \mathscr{L}(\mathscr{X})$ that Sceptic is allowed to pick. If she does, and Reality reveals the outcome $\omega_{n+1} \in \mathscr{X}$, the (possibly negative) amount $\Delta M(\omega_{1:n})(\omega_{n+1})$ goes to Sceptic and her total capital becomes

$$\begin{aligned} M(\omega_{1:n+1}) &= M(\omega_{1:n}) + \Delta M(\omega_{1:n})(\omega_{n+1}) \\ &= M(\square) + \sum_{k=0}^{n-1} \Delta M(\omega_{1:k})(\omega_{k+1}), \end{aligned}$$

with $M(\square)$ her initial capital. By focussing on non-negative supermartingales, we additionally prevent Sceptic from borrowing.

As an important special case, we consider *test supermartingales* $T\colon \mathbb{S} \to \mathbb{R}$ for φ. These are non-negative supermartingales for φ for which $T(\square) := 1$. We collect all test supermartingales for φ in the set $\overline{\mathbb{T}}(\varphi)$. In one of our proofs, we will need a particular way of defining such test supermartingales. To that end, we consider a *multiplier process* D, which is a non-negative gamble process. With every such multiplier process, we associate a non-negative real process $D^{\circledcirc}$ defined by $D^{\circledcirc}(\square) := 1$ and, for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, by the recursion equation $D^{\circledcirc}(sx) := D^{\circledcirc}(s)D(s)(x)$. Given a forecasting system $\varphi \in \Phi$, if a multiplier process D satisfies $\overline{E}_{\varphi(s)}(D(s)) \leq 1$ for all $s \in \mathbb{S}$, then we call D a *supermartingale multiplier* for φ. Every supermartingale multiplier D for φ can be used to construct a test supermartingale for φ.

Proposition 4 *Consider a multiplier process D and a forecasting system φ. If D is a supermartingale multiplier for φ, then $D^{\circledcirc}$ is a test supermartingale for φ.*

Proof of Proposition 4 Since $\overline{E}_{\varphi(s)}(D(s)) \leq 1$ and $D^{\circledcirc}(s) \geq 0$ for all $s \in \mathbb{S}$, it follows from C2 and C4 that

$$\begin{aligned} \overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) &= \overline{E}_{\varphi(s)}\big(D^{\circledcirc}(s\bullet) - D^{\circledcirc}(s)\big) \\ &= \overline{E}_{\varphi(s)}\big(D^{\circledcirc}(s)(D(s)-1)\big) \\ &= D^{\circledcirc}(s)\big(\overline{E}_{\varphi(s)}(D(s)) - 1\big) \leq 0, \end{aligned}$$

for all $s \in \mathbb{S}$. ■

Vice versa, with every positive real process T, we associate a multiplier process D_T, which maps any $s \in \mathbb{S}$ to a gamble $D_T(s)$ defined by

$$D_T(s)(x) := \frac{T(sx)}{T(s)} \text{ for all } x \in \mathscr{X}.$$

Positive supermartingales $T \in \overline{\mathbb{M}}(\varphi)$ have positive supermartingale multipliers D_T for φ.

Proposition 5 *Consider a forecasting system φ and a positive supermartingale T. Then, D_T is a positive supermartingale multiplier for φ.*

Proof of Proposition 5 Clearly, since T is positive, D_T is well-defined and positive. Moreover, since $\overline{E}_{\varphi(s)}(\Delta T(s)) \leq 0$ and $T(s) > 0$ for all $s \in \mathbb{S}$, it follows from C2 and C4 that

$$\begin{aligned} \overline{E}_{\varphi(s)}(D_T(s)) &= \overline{E}_{\varphi(s)}\left(\frac{T(s\bullet)}{T(s)}\right) \\ &= \overline{E}_{\varphi(s)}\left(\frac{T(s) + \Delta T(s)}{T(s)}\right) \\ &= \overline{E}_{\varphi(s)}\left(1 + \frac{\Delta T(s)}{T(s)}\right) \\ &= 1 + \frac{\overline{E}_{\varphi(s)}(\Delta T(s))}{T(s)} \leq 1 \text{ for all } s \in \mathbb{S}, \end{aligned}$$

which completes the proof. ■

4. Computable Betting Strategies

Sceptic will not be allowed to adopt just any supermartingale as a betting strategy. We will also require that it should be computable [1, 3]. Loosely speaking, this means that her betting strategies should be 'describable', in the sense that for each of them there is some finite description that specifies how to approximate it to arbitrary precision. To formalize when a supermartingale is computable, we turn to computability theory.

As a basic building block, this theory considers *recursive* natural functions $\phi: \mathbb{N}_0 \to \mathbb{N}_0$, which are maps that can be computed by a Turing machine [9]. By the Church–Turing thesis, this is equivalent to the existence of a finite algorithm that, given the input $n \in \mathbb{N}_0$, outputs $\phi(n) \in \mathbb{N}_0$. We note that the domain $\mathbb{N}_0$ can be replaced by $\mathbb{N}$, $\mathbb{S}$, $\mathbb{S} \times \mathscr{X}$, $\mathbb{S} \times \mathbb{N}_0$ or any other countably infinite set that can be encoded by a finite alphabet. For example, since a path $\omega \in \Omega$ is a function from $\mathbb{N}$ to $\mathscr{X} = \{0,1\}$, we call it recursive if there is some finite algorithm that, given the input $n \in \mathbb{N}$, outputs $\omega_n \in \mathscr{X}$. Similarly, a selection process S is recursive if there is a finite algorithm that, given the input $s \in \mathbb{S}$, outputs the binary digit $S(s) \in \{0,1\}$.

More generally, for any countable domain $\mathscr{D}$ that can be encoded by a finite alphabet, a rational map $q: \mathscr{D} \to \mathbb{Q}$ is recursive if there are three recursive natural maps $a,b,c: \mathscr{D} \to \mathbb{N}_0$ such that

$$b(d) \neq 0 \text{ and } q(d) = (-1)^{c(d)} \frac{a(d)}{b(d)} \text{ for all } d \in \mathscr{D}.$$

Since a finite number of finite algorithms can always be combined into one finite algorithm [10], this is equivalent to the existence of a finite algorithm that can compute $q(s)$ for all $s \in \mathbb{S}$. In particular, a rational test supermartingale T for φ is recursive if it is recursive as a rational map on $\mathbb{S}$. Moreover, a rational supermartingale multiplier D is recursive if it is recursive as a rational map on $\mathbb{S} \times \mathscr{X}$ that maps any $(s,x) \in \mathbb{S} \times \mathscr{X}$ to $D(s)(x)$.

In what follows, we will need the following relationships between the recursive character of positive rational test supermartingales and positive rational supermartingale multipliers. To prove these (and future) results, we proceed as in [9], and establish a map's recursive character by providing an algorithm for it.

Proposition 6 *Consider any positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ and its associated rational supermartingale multiplier D_T. If T is recursive, then so is D_T.*

Proof of Proposition 6 If T is recursive, then there is a finite algorithm that can compute $T(s)$ for all $s \in \mathbb{S}$. Since the same algorithm can clearly be used to compute $D_T(s)(x) = T(sx)/T(s)$ for all $(s,x) \in \mathbb{S} \times \mathscr{X}$, it follows that D_T is a recursive rational supermartingale multiplier for φ. ∎

Proposition 7 *Consider any rational supermartingale multiplier D for φ and its associated rational test supermartingale $D^{\circledcirc} \in \overline{\mathbb{T}}(\varphi)$. If D is recursive, then so is $D^{\circledcirc}$.*

Proof of Proposition 7 By definition, $D^{\circledcirc}(\square) = 1$ and

$$D^{\circledcirc}(x_{1:n}) = \prod_{k=0}^{n-1} D(x_{1:k})(x_{k+1}),$$

for all $x_{1:n} \in \mathbb{S} \setminus \{\square\}$. Furthermore, since D is recursive, there is a finite algorithm that can compute $D(s)(x)$ for all $(s,x) \in \mathbb{S} \times \mathscr{X}$. Since, for every $x_{1:n} \in \mathbb{S}$, $D^{\circledcirc}(x_{1:n})$ is a finite product of such terms, and since taking the product is a recursive operation, it follows that the same algorithm can be used to compute $D^{\circledcirc}(s)$ for all $s \in \mathbb{S}$. Hence, $D^{\circledcirc}$ is recursive as well. ∎

Computability theory not only considers recursive objects, but also uses them to introduce *computable* ones. The simplest case is that of a computable real number: a real number $x \in \mathbb{R}$ is called computable if there is some recursive rational map $q: \mathbb{N}_0 \to \mathbb{Q}$ such that $|x - q(n)| < 2^{-n}$ for all $n \in \mathbb{N}_0$. More generally, for any countable domain $\mathscr{D}$ that can be encoded by a finite alphabet, a real map $r: \mathscr{D} \to \mathbb{R}$ is called computable if there is a recursive rational map $q: \mathscr{D} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|r(d) - q(d,n)| < 2^{-n}$ for all $d \in \mathscr{D}$ and $n \in \mathbb{N}_0$. Intuitively, a real map is thus computable if there is some finite algorithm that, for every element of the domain, can generate its binary expansion up to any arbitrary precision. In particular, a real process F is computable if there is some recursive rational map $q: \mathbb{S} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|F(s) - q(s,n)| < 2^{-n}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$. Two types of computable processes that we will make frequent use of are computable supermartingales and computable precise forecasting systems.

5. Computable Randomness

Now that we know what a computable supermartingale is, we can finally introduce the martingale-theoretic (imprecise-probabilistic) notion of computable randomness that appears in the title of this contribution. Loosely speaking, a path $\omega \in \Omega$ is computably random for a forecasting system $\varphi \in \Phi$ if there is no computable betting strategy that is allowed by φ and that makes Sceptic arbitrarily rich along ω, without borrowing. This formalizes in the following definition, which we borrow from De Cooman and De Bock [4, Definition 3].

Definition 8 (Computable randomness) *A path $\omega \in \Omega$ is* computably random *for a forecasting system $\varphi \in \Phi$ if there is no computable non-negative real supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ for which $\limsup_{n\to\infty} M(\omega_{1:n}) = +\infty$.*

Interestingly, a path $\omega \in \Omega$ is computably random for a forecasting system $\varphi \in \Phi$ if and only if there is no recursive positive rational betting strategy that starts with unit capital, is allowed by φ and makes Sceptic arbitrarily rich along ω. Since this last notion simplifies our proofs, it is the one that we will use more often.

Proposition 9 ([8, Proposition 4]) *A path $\omega \in \Omega$ is computably random for a forecasting system $\varphi \in \Phi$ if and only if there is no recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ for which $\lim_{n\to\infty} T(\omega_{1:n}) = +\infty$.*

We refer the reader to ([4, 8]) for more information about this imprecise–probabilistic notion of computable randomness, and only mention those results that are relevant to what we want to do here. First of all, we establish that Definition 8 is meaningful, in the sense that every forecasting system $\varphi \in \Phi$ has at least one computably random path.

Proposition 10 ([4, Section 6]) *For every forecasting system $\varphi \in \Phi$, there is at least one path $\omega \in \Omega$ that is computably random for φ.*

Moreover, any $\omega \in \Omega$ that is computably random for $\varphi \in \Phi$ is also computably random for any forecasting system that is less informative—or more conservative—than φ.

Proposition 11 ([4, Proposition 5]) *Let a path $\omega \in \Omega$ be computably random for a forecasting system $\varphi \in \Phi$. Then ω is also computably random for any forecasting system $\varphi^* \in \Phi$ such that $\varphi \sqsubseteq \varphi^*$, meaning that $\varphi(s) \subseteq \varphi^*(s)$ for all $s \in \mathbb{S}$.*

When a forecasting system $\varphi \in \Phi$ uses the very same interval forecast $I \in \mathscr{I}$ in all situations, i.e., $\varphi(s) = I$ for all $s \in \mathbb{S}$, then we call this φ a *stationary* forecasting system, and we simplify the notation by writing I instead of φ. In what follows, we will call a path $\omega \in \Omega$ computably random for an interval forecast $I \in \mathscr{I}$ if it is computably random for the corresponding stationary forecasting system. Interestingly, for any path $\omega \in \Omega$ that is computably random for such an interval forecast $I \in \mathscr{I}$, computable randomness imposes bounds on the relative frequency of ones along all recursively selectable infinite subsequences of ω. That is, ω satisfies an 'imprecise' version of Church randomness.

Proposition 12 ([4, Corollary 11]) *Consider any path $\omega \in \Omega$, any recursive selection process $S \in \mathscr{S}$ for which $\lim_{n\to+\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) = +\infty$, and any interval forecast $I \in \mathscr{I}$. If ω is computably random for I, then*

$$\min I \leq \liminf_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \limsup_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \max I.$$

6. Non-stationary Precise Forecasting Systems versus Interval Forecasts

At this point, we have introduced all the necessary mathematical apparatus for moving towards the main result of this paper; we intend to show that for any rational interval forecast $I \in \mathscr{I}$, there is at least one non-computable non-stationary precise forecasting system $\varphi \in \Phi$ that has the exact same set of computably random paths. To this end, we consider a special class of precise forecasting systems. Fix any two real numbers $p, q \in [0,1]$ and any path $\varpi \in \Omega$. We use these to introduce the forecasting system $\varphi^{\varpi}_{p,q} \in \Phi$, defined by

$$\varphi^{\varpi}_{p,q}(s) := \begin{cases} p & \text{if } \varpi_{|s|+1} = 0 \\ q & \text{if } \varpi_{|s|+1} = 1 \end{cases} \quad \text{for all } s \in \mathbb{S}.$$

We start by observing that if the path ϖ is not recursive, and $p < q$, then the corresponding forecasting system is not computable.

Lemma 13 *Consider any two real numbers $p, q \in [0,1]$ such that $p < q$ and any path $\varpi \in \Omega$. If $\varphi^{\varpi}_{p,q}$ is computable, then ϖ is recursive.*

Proof of Lemma 13 Assume that $\varphi^{\varpi}_{p,q}$ is computable. Consequently, there is some recursive rational map $\tilde{q} \colon \mathbb{S} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|\varphi^{\varpi}_{p,q}(s) - \tilde{q}(s,n)| < 2^{-n}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$. Let's consider the number $\varepsilon := (q-p)/3$. Since $\varepsilon > 0$, we can fix some $N \in \mathbb{N}$ such that $2^{-N} < \varepsilon$. Moreover, let r be a rational number such that $p + \varepsilon < r < q - \varepsilon$; this is always possible because $q - p = 3\varepsilon$. If we consider the recursive rational process F defined by $F(s) := \tilde{q}(s, N)$ for all $s \in \mathbb{S}$, then clearly

$$|\varphi^{\varpi}_{p,q}(s) - F(s)| < 2^{-N} < \varepsilon \quad \text{for all } s \in \mathbb{S}.$$

Consequently, for all $s \in \mathbb{S}$, if $F(s) \leq r$, then $\varphi^{\varpi}_{p,q}(s) < r + \varepsilon < q$, and therefore $\varphi^{\varpi}_{p,q}(s) = p$, and if $r < F(s)$, then $p < r - \varepsilon < \varphi^{\varpi}_{p,q}(s)$, and therefore $\varphi^{\varpi}_{p,q}(s) = q$. Hence, if we consider the trivial path $\omega^0 \in \Omega$ defined by $\omega^0_n := 0$ for all $n \in \mathbb{N}$, then the path ϖ can be inferred from F because

$$\varpi_{n+1} = \begin{cases} 0 & \text{if } F(\omega^0_{1:n}) \leq r \\ 1 & \text{if } F(\omega^0_{1:n}) > r \end{cases} \quad \text{for all } n \in \mathbb{N}_0.$$

Since r is a rational number, ω^0 is a recursive path and F is a recursive rational process, the above inequalities can be checked recursively. Hence, ϖ is recursive. ∎

A sufficient condition for a path ϖ to be non-recursive, is for it to be computably random for an interval forecast I that excludes 0 and 1.

Lemma 14 *Consider any path $\varpi \in \Omega$ and any interval forecast $I \in \mathscr{I}$ for which $0 \notin I$ and $1 \notin I$. If ϖ is recursive, then ϖ is not computably random for I.*

Proof of Lemma 14 Assume that ϖ is recursive. Consequently, the selection processes $S_0, S_1 \in \mathscr{S}$, defined by $S_0(s) := 1 - \varpi_{|s|+1}$ and $S_1(s) := \varpi_{|s|+1}$ for all $s \in \mathbb{S}$, are recursive. Clearly, since ϖ is a binary infinite sequence, it holds that $\lim_{n\to\infty} \sum_{k=0}^{n-1} S_0(\varpi_{1:k}) = \infty$ or $\lim_{n\to\infty} \sum_{k=0}^{n-1} S_1(\varpi_{1:k}) = \infty$, and therefore

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_0(\varpi_{1:k})\varpi_{k+1}}{\sum_{k=0}^{n-1} S_0(\varpi_{1:k})} = 0$$

or

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_1(\varpi_{1:k})\varpi_{k+1}}{\sum_{k=0}^{n-1} S_1(\varpi_{1:k})} = 1.$$

Now, assume *ex absurdo* that ϖ is computably random for I. Consequently, we infer from Proposition 12 that all recursive selection processes $S \in \mathscr{S}$ for which $\lim_{n\to+\infty} \sum_{k=0}^{n-1} S(\varpi_{1:k}) = +\infty$ satisfy

$$0 < \min I \leq \liminf_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(\varpi_{1:k})\varpi_{k+1}}{\sum_{k=0}^{n-1} S(\varpi_{1:k})}$$

and

$$\limsup_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(\varpi_{1:k})\varpi_{k+1}}{\sum_{k=0}^{n-1} S(\varpi_{1:k})} \leq \max I < 1,$$

a contradiction. ■

By combining both results, we arrive at a procedure for obtaining non-computable non-stationary forecasting systems of the form $\varphi^{\varpi}_{p,q}$. Note that, due to Proposition 10, the computably random path ϖ that appears in this construction exists.

Proposition 15 *For any two rational numbers $p,q \in [0,1]$ such that $p < q$, any interval forecast $I \in \mathscr{I}$ for which $0 \notin I$ and $1 \notin I$, and any path $\varpi \in \Omega$ that is computably random for I, the forecasting system $\varphi^{\varpi}_{p,q}$ is non-computable and non-stationary.*

Proof of Proposition 15 Assume *ex absurdo* that $\varphi^{\varpi}_{p,q}$ is computable or stationary. If it is stationary, then since $p \neq q$, it must be that $\varpi_n = 0$ for all $n \in \mathbb{N}$ or $\varpi_n = 1$ for all $n \in \mathbb{N}$. In both cases, ϖ is clearly recursive. The same is true if $\varphi^{\varpi}_{p,q}$ is computable, due to Lemma 13. It therefore follows from Lemma 14 that ϖ is not computably random for I, a contradiction. ■

To gain some intuition about the forecasting systems $\varphi^{\varpi}_{p,q} \in \Phi$ in this result, consider a path $\varpi_{1/2} \in \Omega$ that is computably random for $I = 1/2$. The corresponding forecasting system $\varphi^{\varpi_{1/2}}_{p,q}$ can then be thought of as generated by repeatedly flipping a fair coin: if it lands heads, the subject forecasts p; otherwise, he forecasts q.

In the following theorem, which we consider to be our main result, we use these kinds of forecasting systems to reveal a surprisingly close connection between non-stationary precise forecasting systems and interval forecasts.

Theorem 16 *Consider any two rational numbers $p,q \in [0,1]$ such that $p < q$, any interval forecast $I \in \mathscr{I}$ for which $0 \notin I$ and $1 \notin I$, any path $\varpi \in \Omega$ that is computably random for I, and the corresponding precise forecasting system $\varphi^{\varpi}_{p,q}$. Then a path $\omega \in \Omega$ is computably random for $\varphi^{\varpi}_{p,q}$ if and only if it is computably random for $[p,q]$.*

Proof of Theorem 16 We begin with the direct implication. Assume that $\omega \in \Omega$ is computably random for $\varphi^{\varpi}_{p,q}$. Since $\varphi^{\varpi}_{p,q}(s) \subseteq [p,q]$ for all $s \in \mathbb{S}$, it follows from Proposition 11 that ω is computably random for $[p,q]$.

To prove the converse implication, assume that $\omega \in \Omega$ is computably random for $[p,q]$. To prove that ω is computably random for $\varphi^{\varpi}_{p,q}$, we consider any recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi^{\varpi}_{p,q})$ and prove that it remains bounded along ω. To this end, consider the rational multiplier process D_T. By Proposition 5, D_T is a positive rational supermartingale multiplier for $\varphi^{\varpi}_{p,q}$. Moreover, we know from Proposition 6 that D_T is recursive because T is.

For any $s \in \mathbb{S}$, it cannot happen that both $E_p(D_T(s)) > 1$ and $E_q(D_T(s)) > 1$, since this would violate the inequality $E_{\varphi^{\varpi}_{p,q}(s)}(D_T(s)) \leq 1$, which is a consequence of our assumption that D_T is a supermartingale multiplier for $\varphi^{\varpi}_{p,q}$. Hence, for every $s \in \mathbb{S}$, precisely one of the following cases occurs:

(i) $E_p(D_T(s)) \leq 1$ and $E_q(D_T(s)) > 1$;

(ii) $E_p(D_T(s)) > 1$ and $E_q(D_T(s)) \leq 1$;

(iii) $E_p(D_T(s)) \leq 1$ and $E_q(D_T(s)) \leq 1$;

as is graphically represented in Figure 2.

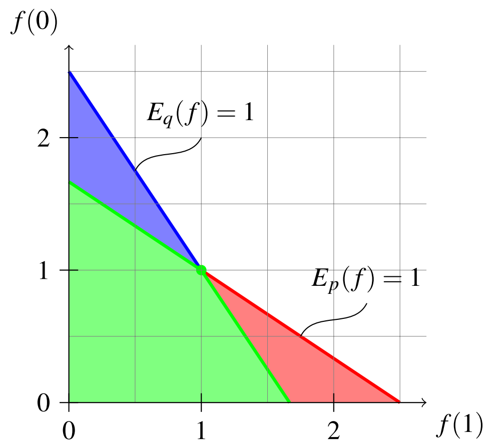

Figure 2: Let $p := 2/5$ and $q := 3/5$. Then the red region depicts all positive gambles $f \in \mathscr{L}(\mathscr{X})$ for which $E_p(f) \leq 1$ and $E_q(f) > 1$, the blue region depicts all positive gambles $f \in \mathscr{L}(\mathscr{X})$ for which $E_p(f) > 1$ and $E_q(f) \leq 1$, and the green region depicts all positive gambles $f \in \mathscr{L}(\mathscr{X})$ for which $E_p(f) \leq 1$ and $E_q(f) \leq 1$.

Consider now three selection processes S_p, S_q and $S_{[p,q]}$ in $\mathscr{S}$, defined for all $s \in \mathbb{S}$ by

$$S_p(s) := \begin{cases} 1 & \text{if } E_p(D_T(s)) \leq 1 \text{ and } E_q(D_T(s)) > 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$S_q(s) := \begin{cases} 1 & \text{if } E_p(D_T(s)) > 1 \text{ and } E_q(D_T(s)) \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$S_{[p,q]}(s) := \begin{cases} 1 & \text{if } E_p(D_T(s)) \leq 1 \text{ and } E_q(D_T(s)) \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then clearly, for each $s \in \mathbb{S}$, exactly one of these processes will have value 1. Going back to Figure 2, $S_p(s) = 1$ indicates that $D_T(s)$ is located in the red region, $S_q(s) = 1$ indicates that $D_T(s)$ is located in the blue region, and $S_{[p,q]}(s) = 1$ indicates that $D_T(s)$ is located in the green region. Moreover, since D_T is a recursive rational process and p and q are rational, it follows from Equation (1) that $E_p(D_T(s))$ and $E_q(D_T(s))$ are rational and recursively determinable for all $s \in \mathbb{S}$. Consequently, the inequalities used for defining S_p, S_q and $S_{[p,q]}$ can be checked in a recursive way, and hence, these selection processes are recursive.

We now take a closer look at the recursive selection process S_p and prove that there is only a finite number of situations $s \in \mathbb{S}$ for which $S_p(s) = 1$. To this end, we consider the selection process $\tilde{S}_p \in \mathscr{S}$ defined by

$$\tilde{S}_p(s) := \begin{cases} 1 & \text{if } S_p(t) = 1 \text{ for some } t \in \mathbb{S} \text{ with } |t| = |s| \\ 0 & \text{if } S_p(t) = 0 \text{ for all } t \in \mathbb{S} \text{ with } |t| = |s|, \end{cases}$$

for all $s \in \mathbb{S}$. Since S_p is recursive and since for every $n \in \mathbb{N}_0$ the situations $t \in \mathbb{S}$ for which $|t| = n$ can be recursively enumerated, the selection process $\tilde{S}_p$ is recursive. Now assume *ex absurdo* that there is an infinite number of situations $s \in \mathbb{S}$ for which $S_p(s) = 1$. Then for all $N \in \mathbb{N}_0$, there is some $s \in \mathbb{S}$ with $|s| > N$ such that $S_p(s) = 1$, and hence, $\lim_{n\to\infty} \sum_{k=0}^{n-1} \tilde{S}_p(\varpi_{1:k}) = \infty$.

It now suffices to prove that

$$\text{for all } s \in \mathbb{S}, \text{ if } \tilde{S}_p(s) = 1, \text{ then } \varpi_{|s|+1} = 0. \qquad (4)$$

Indeed, in that case, since $\lim_{n\to\infty} \sum_{k=0}^{n-1} \tilde{S}_p(\varpi_{1:k}) = \infty$, we will get that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} \tilde{S}_p(\varpi_{1:k})\varpi_{k+1}}{\sum_{k=0}^{n-1} \tilde{S}_p(\varpi_{1:k})} = 0,$$

and therefore, since ϖ was assumed to be random with respect to I, Proposition 12 will imply that $\min I \leq 0$, contradicting the assumption that $0 \notin I$. Hence, we will be able to conclude that there is a finite number of situations $s \in \mathbb{S}$ for which $S_p(s) = 1$.

To prove Statement (4), fix any $s \in \mathbb{S}$ such that $\tilde{S}_p(s) = 1$. This implies that there is some $t \in \mathbb{S}$ with $|t| = |s|$ such that $S_p(t) = 1$. From the definition of S_p, it follows that $E_p(D_T(t)) \leq 1$ and $E_q(D_T(t)) > 1$. Since $E_{\varphi^{\varpi}_{p,q}(t)}(D_T(t)) \leq 1$ (because D_T is a supermartingale multiplier for $\varphi^{\varpi}_{p,q}$), the second of these two inequalities implies that $\varphi^{\varpi}_{p,q}(t) = p$. Since $|t| = |s|$, it follows that indeed $\varpi_{|s|+1} = \varpi_{|t|+1} = 0$.

In a similar way, it can be shown that there are only a finite number of situations $s \in \mathbb{S}$ for which $S_q(s) = 1$.

Next, we consider the positive rational multiplier process $D_{[p,q]}$ defined by

$$D_{[p,q]}(s) := S_{[p,q]}(s)D_T(s) + (1 - S_{[p,q]}(s)) \text{ for all } s \in \mathbb{S}.$$

Since both $S_{[p,q]}$ and D_T are recursive, $D_{[p,q]}$ is recursive as well. Observe that, for all $s \in \mathbb{S}$,

$$D_{[p,q]}(s) = \begin{cases} D_T(s) & \text{if } E_p(D_T(s)) \leq 1 \text{ and } E_q(D_T(s)) \leq 1 \\ 1 & \text{otherwise.} \end{cases}$$

Since $E_p(1) = 1$ and $E_q(1) = 1$ by Equation (1), it follows from Equation (2) that $\overline{E}_{[p,q]}(D_{[p,q]}(s)) \leq 1$ for all $s \in \mathbb{S}$. Hence, $D_{[p,q]}$ is a recursive positive rational supermartingale multiplier for $[p,q]$. Consequently, the real process $D^{\circledcirc}_{[p,q]}$ is a recursive positive rational test supermartingale for $[p,q]$ by Proposition 4 and 7. Since ω is computably random for $[p,q]$ by assumption, $D^{\circledcirc}_{[p,q]}$ must therefore remain bounded along ω, in the sense that $\limsup_{n\to\infty} D^{\circledcirc}_{[p,q]}(\omega_{1:n}) < \infty$.

Now recall that, for all $s \in \mathbb{S}$, exactly one of the processes $S_p(s)$, $S_q(s)$ and $S_{[p,q]}(s)$ will assign the value 1. Since there is only a finite number of situations $s \in \mathbb{S}$ for which $S_p(s) = 1$ or $S_q(s) = 1$, this implies that there is only a finite number of situations $s \in \mathbb{S}$ for which $S_{[p,q]}(s) = 0$, and therefore, only a finite number of situations $s \in \mathbb{S}$ for which $D_T(s) \neq D_{[p,q]}(s)$. For any such situation s, we furthermore have that $D^{\circledcirc}_{[p,q]}(s) = 1$. Consequently, since $\limsup_{n\to\infty} D^{\circledcirc}_{[p,q]}(\omega_{1:n}) < \infty$, it follows that $\limsup_{n\to\infty} T(\omega_{1:n}) < \infty$. Since this is true for any recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$, Proposition 9 implies that ω is computably random for $\varphi^{\varpi}_{p,q}$. ■

What does this result tell us about adopting imprecision in computable randomness?

To start the discussion, we recall from Proposition 11 that if a path $\omega \in \Omega$ is computably random for some precise forecasting system $\varphi \in \Phi$, then it is also computably random for any forecasting system that is less informative. Hence, in particular, for any $\varpi \in \Omega$ and rationals $p < q$, if ω is computably random for $\varphi^{\varpi}_{p,q}$, then it is also computably random for the interval forecast $[p,q]$. So we see that $[p,q]$ can be used as a simpler—yet imprecise—alternative for $\varphi^{\varpi}_{p,q}$. In many cases, this will result in a larger set of computably random paths, and hence, a less informative description of the computable randomness associated with ω. However, interval forecasts do not just serve as an alternative for non-stationary precise forecasts. For example, as showed by De Cooman and De Bock [5, Section 10], there are paths ω that are computably random for $[p,q]$, but not computably random for any precise (possibly non-stationary) computable forecasting system φ. This led them to claim that computable randomness is inherently imprecise, because the randomness of such paths ω can only be captured by an imprecise forecasting system. Theorem 16 shows that the assumption that φ is computable is crucial for this claim, because, for any ϖ that satisfies the conditions of Theorem 16, ω will always be computably random for $\varphi^{\varpi}_{p,q}$, which, as we know from Proposition 15, is not computable. In the next section, we argue that this assumption is justified on practical grounds.

7. Theoretical and Practical Necessity of Interval Forecasts in Statistics

To better understand the implications of Theorem 16, we find it useful to look at it from the point of view of statistics, whose aim it is to learn an uncertainty model—or, equivalently, a forecasting system φ—from an initial segment $\omega_{1:n}$ of an idealised (unobserved) path ω.

It seems justified to require that the path ω is computably random for the forecasting system φ. So let us take a look at the multitude of forecasting systems for which this is the case. From the discussion in [8, Section 5], we know that there is at least one (rational) interval forecast $[p,q]$ that makes ω computably random. Meanwhile, it is not guaranteed that there is a stationary precise forecast p that makes ω computably random. Hence, imprecision is needed if we insist on a stationary description. If we allow for non-stationary uncertainty models however, then Theorem 16 shows that we could replace $[p,q]$ by the non-stationary precise forecasting system $\varphi^{\varpi}_{p,q}$, with ϖ as in Theorem 16. In fact, there is an even easier way to associate a non-stationary precise forecasting system with a path ω.

Proposition 17 *Any path $\omega \in \Omega$ is computably random for the precise forecasting system $\varphi^{\omega}_{0,1}$.*

Proof of Proposition 17 Consider any recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi^{\omega}_{0,1})$. Since T is a supermartingale for $\varphi^{\omega}_{0,1}$, it holds for any $n \in \mathbb{N}_0$ that

$$0 \geq E_{\varphi^{\omega}_{0,1}(\omega_{1:n})}(\Delta T(\omega_{1:n}))$$
$$= \begin{cases} \Delta T(\omega_{1:n})(0) & \text{if } \omega_{n+1} = 0 \\ \Delta T(\omega_{1:n})(1) & \text{if } \omega_{n+1} = 1 \end{cases} = \Delta T(\omega_{1:n})(\omega_{n+1}),$$

and therefore,

$$T(\omega_{1:n}) = T(\square) + \sum_{k=0}^{n-1} \Delta T(\omega_{1:k})(\omega_{k+1}) \leq T(\square) = 1.$$

Consequently, all recursive positive rational test supermartingales $T \in \overline{\mathbb{T}}(\varphi^{\omega}_{0,1})$ are bounded above by 1 along ω. It therefore follows from Proposition 9 that ω is computably random for $\varphi^{\omega}_{0,1}$. ∎

Hence, if ω is computably random for $[p,q]$, then it is also computably random for at least two non-stationary precise models. We won't risk getting bogged down into a discussion on what uncertainty models are best associated with a path ω; that would require a paper on its own. We do want to point out though that the uncertainty models that correspond with ω typically do not contain the same information; that is, they do not share the same set of computably random paths. Interestingly, however, as we know from Theorem 16, $[p,q]$ and $\varphi^{\varpi}_{p,q}$ do have the same set of computably random paths and are, in that sense, equally expressive. On that ground, theoretically, one might argue that imprecision is not needed.

We believe that this story changes when moving to more practical grounds. Imagine that we are given an initial finite segment $\omega_{1:n}$ of a path $\omega \in \Omega$, and that we want to learn a forecasting system φ for which ω is computably random. We will have to do so by adopting a finite algorithm that, given the data $\omega_{1:n}$, outputs a forecasting system φ' whose set of computably random paths is then believed to contain ω. A candidate for φ' could be the forecasting system $\varphi^{\omega}_{0,1}$ that is generated by ω itself. However, it seems impossible to learn this forecasting system, as it basically requires us to know ω itself. Another candidate for φ' could be $\varphi^{\varpi}_{p,q}$. Here too, however, it seems impossible to learn this model because it is non-computable. Meanwhile, we believe that our chances of learning the equally expressive interval forecast $[p,q]$ are much higher.

In summary, it is one thing to associate precise uncertainty models with a path ω, but it is another thing to actually learn them. When it comes to the latter, imprecise forecasts seem more promising than non-computable non-stationary precise ones.

8. Conclusions and Future Work

We conclude that rational interval forecasts are actually quite precise, in the sense that they have the same set of computably random paths as the related non-computable non-stationary precise forecasting systems $\varphi^{\varpi}_{p,q}$, meanwhile being simpler, stationary and computable. Moreover, our analysis suggests that the computable character of rational interval forecasts will be of utmost importance when moving to the field of statistics. In particular, it seems possible nor opportune to try and learn—or even approximate—non-computable non-stationary precise forecasting systems, which—by definition—cannot be described by a finite algorithm, from a finite initial path segment $\omega_{1:n}$.

In our future work, we plan to investigate if our results apply to other notions of randomness as well, such as Martin-Löf randomness, Schnorr randomness and Church randomness. We also plan to further investigate our preliminary ideas about a randomness-based approach to statistics, and try to develop new statistical methods based on them. Last, we plan to continue exploring the marvellous differences and similarities between classical and imprecise-probabilistic notions of randomness.

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