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Basic Probability Assignments Representable via Belief Intervals for Singletons in Dempster-Shafer Theory

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Abstract

Dempster-Shafer Theory (DST) or Evidence theory has been commonly employed in the literature to deal with uncertainty-based information. The basis of this theory is the concept of basic probability assignment (BPA). The belief intervals for singletons obtained from a BPA have recently received considerable attention for quantifying uncertainty in DST. Indeed, they are easier to manage than the corresponding BPA to represent uncertainty-based information. Nonetheless, the set of probability distributions consistent with a BPA is smaller than the one compatible with the corresponding belief intervals for singletons. In this research, we give a new characterization of BPAs representable by belief intervals for singletons. Such a characterization might be easier to check than the one provided in previous works. In practical applications, this result allows efficiently knowing when uncertainty can be represented via belief intervals for singletons rather than the associated BPA without loss of information.

Keywords: Dempster-Shafer theory, uncertainty-based information, belief intervals, basic probability assignment, focal element

1. Introduction

Dempster-Shafer theory (DST), also known as *Evidence theory* [8, 20], has been frequently employed in the literature to deal with uncertainty-based information in some domains such as *medical diagnosis* [3], *target identification* [4], *face recognition* [13], or *statistical classification* [10]. This theory has also been widely used in the combination of information provided by different sources [2, 12, 5], an important issue for decision making.

The basis of DST is the concept of *basic probability assignment* (BPA), which generalizes the probability distribution concept in Probability Theory (PT). For each BPA in DST, there is a lower and upper probability function associated with it. They are called, respectively, *belief* and *plausibility* functions. In DST, the quantification of the uncertainty-based information represented by a BPA is an essential point. For this purpose, many uncertainty measures in DST have been proposed so far. Some of the most recent researches on this topic can be found in [14, 19, 6, 15].

Most of the uncertainty measures proposed so far in DST are based on BPAs. Nevertheless, the belief intervals composed of the belief and plausibility values of the singletons have recently received considerable attention for calculating the uncertainty-based information represented by a BPA [9, 23, 21, 17]. As explained by Sun et al. [21], Moral-García and Abellán [17], belief intervals are easier to manage than a BPA to represent uncertainty-based information; they belong to reachable probability intervals theory [7]. Belief intervals have also been recently used for the combination of information from different sources [21].

However, the set of probability distributions compatible with the belief intervals for singletons is larger than the one associated with the corresponding BPA [17]. In consequence, when the belief intervals for singletons are used instead of the BPA, some information might be lost. A necessary and sufficient condition under which a BPA and its corresponding set of belief intervals for singletons represent the same uncertainty-based information was given by Moral-García and Abellán [18]. It is expressed in terms of the belief function associated with the BPA. We must remark that such a condition might be computationally hard to check since it requires calculating the belief value for each subset, and the number of subsets exponentially grows as the number of alternatives increases.

For this reason, in this research, we provide a new characterization of BPAs representable via their corresponding set of belief intervals for singletons. Such a characterization is given through the relations among the cardinalities of the subsets for which the probability mass assigned by the BPA is greater than 0 (focal elements of the BPA). Hence, it may be easier to check than the characterization provided in previous works. In practical applications, this result allows efficiently knowing when uncertainty can be represented via belief intervals for singletons rather than the associated BPA without loss of information.

This paper is arranged as follows: Dempster-Shafer theory is exposed in Section 2. Section 3 describes basic probability assignments representable via belief intervals for singletons. Conclusions and plans for future work are given in Section 4.

2. Dempster-Shafer Theory

Let us consider a finite set of possible alternatives $X = \{x_1, x_2 \ldots, x_n\}$. Let $\wp(X)$ be the power set of X.

Dempster-Shafer theory (DST), or Evidence theory [8, 20], is based on the concept of *basic probability assignment*, which consists of a function $m : \wp(X) \to [0,1]$ such that $m(\emptyset) = 0$ and $\sum_{A \subseteq X} m(A) = 1$.

If $A \subseteq X$ satisfies that $m(A) > 0$, A is said to be a *focal element* of m.

A given BPA m on X has a belief function Bel_m, and a plausibility function Pl_m, associated with it. They are defined, for each $A \subseteq X$, as follows:

$$Bel_m(A) = \sum_{B \mid B \subseteq A} m(B), \quad Pl_m(A) = \sum_{B \mid B \cap A \neq \emptyset} m(B). \quad (1)$$

Obviously, $Bel_m(A) \leq Pl_m(A)\ \forall A \subseteq X$. The interval $[Bel_m(A), Pl_m(A)]$ is known as the *belief interval* of A.

In addition,

$$Pl_m(A) = 1 - Bel_m(\overline{A}) \quad \forall A \subseteq X, \quad (2)$$

being $\overline{A}$ the complementary set of A.

Given a BPA m on X, there exists a set of probability distributions consistent with it (really, closed and convex set of probability distributions, also called credal set). It is determined by:

$$\mathscr{P}_m = \{p \in \mathscr{P}(X) \mid Bel_m(A) \leq p(A), \quad \forall A \subseteq X\}, \quad (3)$$

where $\mathscr{P}(X)$ is the set of all probability distributions on X.

We may note that $Bel_m(A) \leq p(A)\ \forall A \subseteq X$ is equivalent to $Bel_m(A) \leq p(A) \leq Pl_m(A)\ \forall A \subseteq X$ due to the duality relation expressed in Equation (2).

3. BPAs Representable by Belief Intervals for Singletons

3.1. Previous Characterization

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of possible alternatives and m a BPA on X. Let Bel_m be the belief function associated with m and Pl_m its corresponding plausibility function.

Let us consider the set of belief intervals for singletons:

$$\mathscr{I}_m = \{[Bel_m(\{x_i\}), Pl_m(\{x_i\})], \quad i = 1, 2, \ldots, n\}. \quad (4)$$

This set of intervals is coherent [17], and the credal set composed of all the probability distributions consistent with these intervals is given by:

$$\mathscr{P}(\mathscr{I}_m) = \{p \in \mathscr{P}(X) \mid Bel_m(\{x_i\}) \leq p(\{x_i\}) \leq Pl_m(\{x_i\}), \quad \forall i = 1, \ldots, n\}.$$

Let $\mathscr{P}_m$ be the credal set corresponding to m:

$$\mathscr{P}_m = \{p \in \mathscr{P}(X) \mid Bel_m(A) \leq p(A), \quad \forall A \subseteq X\}. \quad (5)$$

We may observe that the BPA m and its corresponding set of belief intervals for singletons $\mathscr{I}_m$ represent the same information if, and only if, $\mathscr{P}(\mathscr{I}_m) = \mathscr{P}_m$.

The set of probability distributions compatible with $\mathscr{I}_m$ always contains the credal set associated with m, i.e $\mathscr{P}_m \subseteq \mathscr{P}(\mathscr{I}_m)$ [17].

Nonetheless, the credal set corresponding to m does not always coincide with the one compatible with the belief intervals for singletons, as shown with the following example by Abellán [1], Moral-García and Abellán [18]:

Example 1 *Let $X = \{x_1, x_2, x_3, x_4\}$ be a finite set and m the following BPA on X:*

$$m(\{x_1, x_2\}) = m(\{x_3, x_4\}) = 0.5.$$

Let Bel_m and Pl_m denote, respectively, the belief and plausibility functions associated with m. We have that:

$$Bel_m(\{x_i\}) = m(\{x_i\}) = 0,$$
$$Pl_m(\{x_i\}) = \sum_{A \mid x_i \in A} m(A) = 0.5, \quad \forall 1 \leq i \leq 4.$$

Thus, we have the following set of belief intervals for singletons:

$$\mathscr{I}_m = \{[0, 0.5]; [0, 0.5]; [0, 0.5]; [0, 0.5]\}.$$

Let us consider the probability distribution given by $p = (p(\{x_1\}), p(\{x_2\}), p(\{x_3\}), p(\{x_4\})) = (0.5, 0.5, 0, 0)$. This probability distribution is consistent with $\mathscr{I}_m$. However, it does not belong to the credal set corresponding to m because

$$p(\{x_3, x_4\}) = 0 < 0.5 = m(\{x_3, x_4\}) = Bel_m(\{x_3, x_4\}).$$

Consequently, in this case, the credal set associated with $\mathscr{I}_m$ and the one corresponding to m are not equal.

According to the following result, proved by Moral-García and Abellán [18], the BPA m represents the same information as its associated set of belief intervals for singletons if, and only if, the corresponding belief function is obtained as in reachable probability intervals theory.

Theorem 1

$$\mathscr{P}(\mathscr{I}_m) = \mathscr{P}_m \Leftrightarrow Bel_m(A) = \max\left\{\sum_{x_i \in A} Bel_m(\{x_i\}), 1 - \sum_{x_i \notin A} Pl_m(\{x_i\})\right\}, \quad \forall A \subseteq X.$$

3.2. New Characterization

The condition given in Theorem 1 might be computationally hard to check since it requires computing the value of the belief function for each $A \subseteq X$, and the number of subsets exponentially grows as the number of alternatives increases.

For this reason, in this work, we aim to provide a necessary and sufficient condition for m to represent the same information as $\mathscr{I}_m$ in terms of the relations among the focal elements of m.

We transform m into the following BPA m' on X:

$$m'(\{x_i\}) = 0, \quad \forall i = 1,2,\dots,n,$$
$$m'(A) = \frac{m(A)}{1-M}, \quad \forall A \subseteq X,\, |A| \geq 2,$$

where M = $\sum_{i=1}^{n} m(\{x_i\})$[1].

The following proposition shows that m' is well-defined as a BPA on X.

Proposition 2 *m' is a BPA on X.*

Proof Since m is a BPA on X, it holds that:

$$m(A) \geq 0 \wedge 1-M \geq 0 \Rightarrow m'(A) \geq 0 \quad \forall A \subseteq X, |A| \geq 2,$$

$$m(A) + M \leq \sum_{B\subseteq X} m(B) = 1 \Rightarrow m(A) \leq 1-M \Rightarrow$$
$$m'(A) \leq 1 \quad \forall A \subseteq X,\, |A| \geq 2,$$

$$\sum_{A\subseteq X} m'(A) = \sum_{A\subseteq X,\,|A|\geq 2} \frac{m(A)}{1-M} = \frac{1-\sum_{i=1}^{n} m(\{x_i\})}{1-M} = 1.$$

■

All the focal elements of m' have a cardinality greater or equal than 2. Among the non-singleton subsets, the focal elements of m' coincide with the ones of m. It is expressed in the following result, whose proof is immediate.

Proposition 3 *$\forall A \subseteq X$ such that $|A| \geq 2$, $m(A) > 0 \Leftrightarrow m'(A) > 0$.*

Let $\mathscr{P}_{m'}$ denote the credal set consistent with m', $\mathscr{P}(\mathscr{I}_{m'})$ the credal set compatible with the belief intervals associated with m' for singletons, and $Bel_{m'}$ and $Pl_{m'}$ the belief and plausibility functions corresponding to m', respectively. The following proposition shows that m represents the same information as its associated belief intervals for singletons if, and only if, the same occurs with m'.

Proposition 4 *$\mathscr{P}(\mathscr{I}_m) = \mathscr{P}_m \Leftrightarrow \mathscr{P}(\mathscr{I}_{m'}) = \mathscr{P}_{m'}$.*

1. Here, we do not consider the case $M = 1$ because, in this situation, m is a probability distribution.

Proof For each $A \subseteq X$, we have that:

$$Bel_{m'}(A) = \max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} \Leftrightarrow$$

$$\sum_{B\subseteq A} m'(B) = \max\left\{\sum_{x_i\in A} m'(\{x_i\}), 1 - \sum_{x_i\notin A}\sum_{B|x_i\in B} m'(B)\right\} \Leftrightarrow$$

$$\sum_{B\subseteq A,\,|B|\geq 2} \frac{m(B)}{1-M} = \max\left\{0, 1 - \sum_{x_i\notin A}\sum_{B|x_i\in B\wedge|B|\geq 2} \frac{m(B)}{1-M}\right\} \Leftrightarrow$$

$$\sum_{B\subseteq A,\,|B|\geq 2} m(B) = \max\left\{0, 1 - M - \sum_{x_i\notin A}\sum_{B|x_i\in B\wedge|B|\geq 2} m(B)\right\} \Leftrightarrow$$

$$\sum_{x_i\in A} m(\{x_i\}) + \sum_{B\subseteq A,\,|B|\geq 2} m(B) = \max\left\{\sum_{x_i\in A} m(\{x_i\}),\right.$$
$$\left. 1 - M + \sum_{x_i\in A} m(\{x_i\}) - \sum_{x_i\notin A}\sum_{B|x_i\in B\wedge|B|\geq 2} m(B)\right\} \Leftrightarrow$$

$$\sum_{B\subseteq A} m(B) = \max\left\{\sum_{x_i\in A} Bel_m(\{x_i\}), 1 - \right.$$
$$\left.\sum_{x_i\notin A}\left[m(\{x_i\}) + \sum_{B|x_i\in B\wedge|B|\geq 2} m(B)\right]\right\} \Leftrightarrow$$

$$Bel_m(A) = \max\left\{\sum_{x_i\in A} Bel_m(\{x_i\}), 1 - \sum_{x_i\notin A}\sum_{B|x_i\in B} m(B)\right\} \Leftrightarrow$$

$$Bel_m(A) = \max\left\{\sum_{x_i\in A} Bel_m(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_m(\{x_i\})\right\},$$

and our thesis follows from Theorem 1.

■

Hence, we focus on studying when m' can be represented via its corresponding set of belief intervals for singletons.

The following theorem gives the necessary and sufficient condition for m' to represent the same uncertainty-based information as its corresponding set of belief intervals for singletons: the difference between each pair of focal elements has a cardinality lower than 2.

Theorem 5 *$\mathscr{P}(\mathscr{I}_{m'}) = \mathscr{P}_{m'} \Leftrightarrow |B_1 \setminus B_2| \leq 1\ \forall B_1, B_2 \subseteq X$ such that $m'(B_1) > 0$ and $m'(B_2) > 0$.*

Proof Let us suppose that $\mathscr{P}(\mathscr{I}_{m'}) = \mathscr{P}_{m'}$. Let A be a focal element of m'. From Theorem 1, it follows that:

$$\begin{aligned} Bel_{m'}(A) &= \max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} \\ &= \max\left\{\sum_{x_i\in A} m'(\{x_i\}), 1 - \sum_{x_i\notin A}\sum_{B|x_i\in B} m'(B)\right\} \\ &= \max\left\{0, 1 - \sum_{B\subseteq X} m'(B)\,|B\setminus A|\right\} \\ &= 1 - \sum_{B\subseteq X} m'(B)\,|B\setminus A|. \end{aligned}$$

The last equality is because A is a focal element of m'. In consequence, $\sum_{B\subseteq A} m'(B) = 1 - \sum_{B\subseteq X} m'(B)\,|B\setminus A|$. Furthermore,

$$1 = \sum_{B\subseteq X} m'(B) = \sum_{B\subseteq A} m'(B) + \sum_{B|B\setminus A\neq\emptyset} m'(B).$$

Thus,

$$\sum_{B\subseteq A} m'(B) = 1 - \sum_{B\subseteq X} m'(B)\,|B\setminus A| = \sum_{B\subseteq A} m'(B) + \sum_{B|B\setminus A\neq\emptyset} m'(B) - \sum_{B\subseteq X} m'(B)\,|B\setminus A|.$$

This implies that

$$\sum_{B|B\setminus A\neq\emptyset} m'(B) = \sum_{B\subseteq X} m'(B)\,|B\setminus A|.$$

Therefore, if $m'(B) > 0$, then it is not possible that $|B\setminus A| \geq 2$. It can be concluded that $|B_1\setminus B_2| \leq 1\ \forall B_1, B_2 \subseteq X$ such that $m'(B_1) > 0$ and $m'(B_2) > 0$.

Let us assume now that $|B_1\setminus B_2| \leq 1\ \forall B_1, B_2 \subseteq X$ such that $m'(B_1) > 0$ and $m'(B_2) > 0$. Let us consider $A\subseteq X$. We have that:

$$\max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} = \max\left\{\sum_{x_i\in A} m'(\{x_i\}), 1 - \sum_{x_i\notin A}\sum_{B|x_i\in B} m'(B)\right\} = \max\left\{0, 1 - \sum_{B\subseteq X} m'(B)\,|B\setminus A|\right\} = \max\left\{0, \sum_{B\subseteq A} m'(B) + \sum_{B|B\setminus A\neq\emptyset} m'(B) - \sum_{B\subseteq X} m'(B)\,|B\setminus A|\right\}.$$

We distinguish two cases:

- **Case 1:** $\exists C\subseteq A$ **such that** $m'(C) > 0$.

  By hypothesis, it holds that, if $B\subseteq X$ satisfies that $|B\setminus C| > 1$, then $m'(B) = 0$. Hence, since $C\subseteq A$, $m'(B) = 0\quad \forall B\subseteq X$ such that $|B\setminus A| > 1$. Consequently, $|B\setminus A| \leq 1\ \forall B\subseteq X$ such that $m'(B) > 0$. Then,

$$\max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} = \max\left\{0, \sum_{B\subseteq A} m'(B) + \sum_{B|B\setminus A\neq\emptyset} m'(B) - \sum_{B\subseteq X} m'(B)\,|B\setminus A|\right\} = \max\left\{0, \sum_{B\subseteq A} m'(B)\right\} = \sum_{B\subseteq A} m'(B) = Bel_{m'}(A).$$

- **Case 2:** $m'(B) = 0\ \forall B\subseteq A$.

  In this case, $Bel_{m'}(A) = 0$ and

$$\max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} = \max\left\{0, \sum_{B\subseteq A} m'(B) + \sum_{B|B\setminus A\neq\emptyset} m'(B) - \sum_{B\subseteq X} m'(B)\,|B\setminus A|\right\} = \max\left\{0, \sum_{B|B\setminus A\neq\emptyset} m'(B)\,(1 - |B\setminus A|)\right\} = 0 = Bel_{m'}(A).$$

In this way,

$$\max\left\{\sum_{x_i\in A} Bel_{m'}(\{x_i\}), 1 - \sum_{x_i\notin A} Pl_{m'}(\{x_i\})\right\} = Bel_{m'}(A), \quad \forall A\subseteq X,$$

and, from Theorem 1, we conclude that $\mathscr{P}(\mathscr{I}_{m'}) = \mathscr{P}_{m'}$. ■

As a consequence of this theorem and Proposition 3, the necessary and sufficient condition for a given BPA on X to be representable by its associated set of belief intervals for singletons is the following one: the cardinality of the difference between each pair of non-singleton focal elements is lower or equal than 1. It is expressed in the following corollary:

Corollary 6 *Given a BPA m on a finite set X, it holds that $\mathscr{P}(\mathscr{I}_m) = \mathscr{P}_m \Leftrightarrow |B_1\setminus B_2| \leq 1 \quad \forall B_1, B_2\subseteq X$ such that $m(B_i) > 0$ and $|B_i| \geq 2$, for $i = 1,2$.*

Hence, the BPA m given in Example 1 cannot be represented via its corresponding set of belief intervals for singletons since $\{x_1,x_2\}$ and $\{x_3,x_4\}$ are focal elements of m and $|\{x_3,x_4\}\setminus\{x_1,x_2\}| = 2$.

We show below another example where the BPA is representable by its associated set of belief intervals for singletons.

Example 2 *Let $X = \{x_1,x_2,x_3,x_4\}$ be a finite set and m the following BPA on X:*

$$m(\{x_3\}) = 0.3, \quad m(\{x_1,x_2\}) = 0.3, \quad m(\{x_1,x_2,x_3\}) = 0.1, \quad m(\{x_1,x_2,x_4\}) = 0.3.$$

The non-singleton focal elements of m are $\{x_1,x_2\}$, $\{x_1,x_2,x_3\}$, and $\{x_1,x_2,x_4\}$. We can check that $|\{x_1,x_2\}\setminus\{x_1,x_2,x_i\}| = 0$, $|\{x_1,x_2,x_i\}\setminus\{x_1,x_2\}| = 1$, and $\left|\{x_1,x_2,x_i\}\setminus\{x_1,x_2,x_j\}\right| = 1$, for $i,j\in\{3,4\}$.

Thus, in this case, the credal set corresponding to m coincides with the one associated with the set of belief intervals for singletons.

If there is a unique non-singleton focal element of m, namely B, then $m'(B) = 1$ and, clearly, m y m' can be represented via their corresponding sets of belief intervals for singletons. Also, if all the focal elements of m are singletons, then m is a probability distribution.

For testing the condition given in Corollary 6, it is just necessary to check whether there exists, among the non-singletons subsets, a focal element of greatest cardinality such that its difference with another one of smallest cardinality has a cardinality greater than one. So, that condition might be easier to check than the one given in Theorem 1.

From Corollary 6, it is easy to deduce that, if there are three or fewer alternatives, then m always represents the same uncertainty-based information as its associated set of belief intervals for singletons. Therefore, we have the following result:

Corollary 7 *Let m be a BPA on a finite set $X = \{x_1, \ldots, x_n\}$ with $n \leq 3$. Then, it is always satisfied that $\mathscr{P}(\mathscr{I}_m) = \mathscr{P}_m$.*

3.3. An Example: The Imprecise Dirichlet Model

The Imprecise Dirichlet Model (IDM) [22] is an imprecise probabilities model that is useful to make inferences about the probability distribution of a discrete variable.

Summarizing, the IDM can be described in the following way: Let X be a categorical variable that takes values in $\{x_1, x_2, \ldots, x_n\}$. Let us assume that we have a sample of N independent and identically distributed outcomes of X.

According to the IDM, the probability that X takes the x_i value belongs to the following interval:

$$p(x_i) \in \left[\frac{n_i}{N+s}, \frac{n_i+s}{N+s}\right], \tag{6}$$

where n_i is the number of ocurrences of x_i in the sample $\forall i = 1, 2, \ldots, n$, and $s > 0$ is a given parameter of the model.

As shown by Abellán [1], Denœux [11], the IDM can also be expressed via the following BPA on $\{x_1, x_2, \ldots, x_n\}$:

$$\begin{aligned}
&m(\{x_i\}) = \frac{n_i}{N+s}, \quad \forall i = 1, 2, \ldots, n, \\
&m(A) = 0, \quad \forall A \subseteq \{x_1, x_2, \ldots, x_n\}, 2 \leq |A| < n, \\
&m(\{x_1, x_2, \ldots, x_n\}) = \frac{s}{N+s}.
\end{aligned}$$

In this way, the only focal element of the IDM BPA with cardinality greater than 1 is the total set. Therefore, the IDM BPA can be represented by its corresponding set of belief intervals for singletons. Indeed, these intervals coincide with the ones given in Equation (6).

4. Conclusions and Future Work

Belief intervals for singletons are easier to manage than basic probability assignments to represent uncertainty-based information in Dempster-Shafer theory. Nevertheless, the set of probability distributions corresponding to a BPA is smaller than the one consistent with the corresponding belief intervals for singletons. In this research, we have provided a new characterization of BPAs representable via belief intervals for singletons.

Specifically, the results presented in this work have revealed the necessary and sufficient condition for a BPA to be representable by its associated set of belief intervals for singletons: the cardinality of the difference between each pair of non-singleton focal elements is lower or equal than one. This condition might be easier to check than the one provided in previous works, which requires calculating the belief value for each subset.

It is remarkable that, in DST, when there are three or fewer alternatives, a BPA can always be represented by belief intervals for singletons. Moreover, as an example, we have shown that the Imprecise Dirichlet Model is representable via a BPA such that the set of probability distributions associated with it coincides with the one compatible with the corresponding belief intervals for singletons.

As future work, the results presented here could be used in practical applications to know efficiently when the uncertainty can be represented via belief intervals for singletons rather than the BPA without loss of information. For instance, our results could be applied to the management of uncertainty-based information in sensors [16]. Furthermore, it would be interesting to analyze whether, in other mathematical models expressed by a BPA, the set of probability distributions compatible with it is the same as the one corresponding to the belief intervals for singletons.

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