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Quantum Indistinguishability through Exchangeable Desirable Gambles

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Abstract

Two particles are identical if all their intrinsic properties, such as spin and charge, are the same, meaning that no quantum experiment can distinguish them. In addition to the well known principles of quantum mechanics, understanding systems of identical particles requires a new postulate, the so called *symmetrization postulate*. In this work, we show that the postulate corresponds to exchangeability assessments for sets of observables (gambles) in a quantum experiment, when quantum mechanics is seen as a normative and algorithmic theory guiding an agent to assess her subjective beliefs represented as (coherent) sets of gambles. Finally, we show how sets of exchangeable observables (gambles) may be updated after a measurement and discuss the issue of defining entanglement for indistinguishable particle systems.

Keywords: quantum theory, indistinguishable particles, exchangeability, desirable gambles.

1. Introduction

In recent works (Benavoli et al., 2016, 2017) and in particular in (Benavoli et al., 2019b), we defined a theory of probability on a continuous space of complex vectors that complies with the two postulates of coherence ("The theory should be logically consistent"), and of computation ("Inferences in the theory should be computable in polynomial time"). We then showed that its deductive closure is tantamount to Quantum Mechanics (QM). Hence QM may be viewed as a normative and algorithmic theory guiding an agent to assess her subjective beliefs represented as (coherent) sets of gambles on the results of a quantum experiment. We were then able to derive (in a coherent way) the main postulates of QM from standard operations in probability theory (updating, marginalisation, time coherence). This means we derived a theory of probability which theoretically and empirically agrees with QM experiments.

When one considers systems including more than one particle, we must consider the implications of another important empirical observation: in many of these systems the particles of interest belong to distinct classes of indistinguishable (identical) particles. Two (or more) particles are said to be identical if all their properties (charge, mass, spin, etc.) are exactly the same. In other words, no experiment can distinguish one from the other. Hence, all the electrons in the universe are identical, as are all the protons. This means that, when a physical system contains two identical particles, there is no change in its properties if the roles of these two particles are exchanged.

This law is formulated in QM by the *symmetrization postulate*, which establishes that in a system containing identical particles the only possible configurations of their properties (e.g., spin) are either all symmetrical or all antisymmetrical with respect to permutations of the labels of the particles. In the first case, the particles are called *bosons*; in the second case they are called *fermions*.

In this paper, we aim to derive the *symmetrization postulate* from the way a subject accepts gambles on experiments involving indistinguishable particles. We assume that the particles are *exchangeable*, meaning roughly that the subject believes that the labels (i.e. electron 1, electron 2,..) we use to denote them, has no influence on the decisions and inferences she will make regarding the particles.

Exchangeability is a fundamental concept in classical probability theory and statistics (Diaconis and Freedman, 1980; Regazzini, 1991). Its assumption, and the analysis of its consequences, goes back to de Finetti (1974–1975) and his famous *Representation Theorem*. In statistics, this theorem is interpreted as stating that "a sequence of random variables is exchangeable if it is conditionally independent and identically distributed." This theorem was generalised to QM by Caves et al. (2002) for *quantum-state tomography*, which is a technique to estimate the density matrix of a particle by performing repeated measures (the order of the measures is assumed to be exchangeable).

In this paper, we instead deal with the exchangeability of indistinguishable particles. We show that we can derive the *symmetrization postulate* by using the general framework for exchangeable gambles proposed by De Cooman and Quaeghebeur (2012) for classical (imprecise) probability theory.[1] This confirms, once again, that QM is a subjective theory of probability.

1. Exchangeability in the context of imprecise probability was originally proposed by (Walley, 1991, Sec. 9.5)

The rest of the paper is organised as follows. In Section 2 we recall how QM can be formulated, and thus understood, as an algorithmic theory of desirable gambles. After formulating in Section 3 the symmetrisation postulate, in the following Section 4 we derive it in terms of (algorithmic) coherence and exchangeability. Finally, in Section 5 we show how sets of exchangeable observables (gambles) may be updated after a measurement and in Section 6 we discuss the issue of defining entanglement for indistinguishable particle systems

2. Algorithmic Rationality and QM

In this section, we recall some definition and results from (Benavoli et al., 2019b). Consider a systems of m particles (each one is an n_j-level system, for instance if we consider the spin of an electron $n_j = 2$: the spin can be "up" or "down"). When $m > 1$ the system is said to be composite, whereas in case $m = 1$ we are considering a single particle system. Hence, the possibility space is

$$\Omega = \times_{j=1}^{m} \overline{\mathbb{C}}^{n_j}.$$

where

$$\overline{\mathbb{C}}^{n_j} = \{x \in \mathbb{C}^{n_j} : x^\dagger x = 1\}.$$

Next, we describe the observables, the gambles in our setting. Let us recall that in QM any real-valued observable is described by a Hermitian operator (matrix). This naturally imposes restrictions on the type of 'permitted gambles' g on a quantum experiment. For a single particle, given a Hermitian operator $G \in \mathscr{H}^{n\times n}$ (with $\mathscr{H}^{n\times n}$ being the set of Hermitian matrices of dimension $n \times n$), a gamble on $x \in \overline{\mathbb{C}}^n$ can be defined as:

$$g(x) = x^\dagger G x.$$

Since G is Hermitian and x is bounded ($x^\dagger x = 1$), g is a real-valued bounded function. For a composite system of m particles, the gambles are m-quadratic forms:

$$g(x_1, \ldots, x_m) = (\otimes_{j=1}^{m} x_j)^\dagger G (\otimes_{j=1}^{m} x_j), \tag{1}$$

with $G \in \mathscr{H}^{n\times n}$, $n = \prod_{j=1}^{m} n_j$, and where $\otimes$ denotes the tensor product between vectors regarded as column matrices.[2] Therefore, we have that

$$\mathscr{L}_R = \{g \mid \; G \in \mathscr{H}^{n\times n}\}$$

is the restricted set of 'permitted gambles' in a quantum experiment. We can also define the subset of nonnegative gambles $\mathscr{L}_R^{\geq} := \{g \in \mathscr{L}_R \mid \min g \geq 0\}$ and the subset of negative gambles $\mathscr{L}_R^{<} := \{g \in \mathscr{L}_R \mid \max g < 0\}$.[3]

Since $\mathscr{L}_R$ is a vector space including the constant gambles ($G = cI$ with I identity matrix),[4] we can use standard desirability to impose rationality principles (coherence) in the way a subject should accept gambles. However, this would not lead to QM. Indeed, as discussed in the Introduction, QM follows by the two principles of coherence and of computation.[5]

As shown by Gurvits (2003), for $m > 1$ the problem of deciding whether a gamble is nonnegative, that is whether it belongs to $\mathscr{L}_R^{\geq}$, is NP-hard, thus leading to a violation of the aforementioned computation principle.[6] To fulfil the computation requirement, we therefore need to change the meaning of 'being nonnegative' by considering a subset $\Sigma^{\geq} \subsetneq \mathscr{L}^{\geq}$ for which the membership problem is in P. This is done by considering the following new set of "tautologies":

$$\Sigma^{\geq} := \{g \in \mathscr{L}_R \mid G \geq 0\}.$$

That is, a gamble is 'nonnegative' whenever G is PSD. Note that $\Sigma^{\geq}$ is the so-called cone of *Hermitian sum-of-squares* polynomials.

What described above is the essence of the algorithmic rationality behind QM. In other words, the corresponding algorithmic theory of desirable gambles is based on the following redefinition of the tautologies:

- $\Sigma^{\geq}$ should always be desirable,

The rest of the theory follows exactly the footprints of the standard theory of (almost) desirability. In particular, the deductive closure for a finite[7] set of assessment $\mathscr{G}$ is defined by: [8]

- $\mathscr{C} := \text{posi}(\Sigma^{\geq} \cup \mathscr{G})$.

And finally the coherence postulate simply states that

- A set $\mathscr{C}$ of desirable gambles is said to be *A-coherent* if and only if $-1 \notin \mathscr{C}$,

where 'A' stands for the the fact that the algorithmic bounds of the coherence problem for a finite set of assessments are established according to the choice of $\Sigma^{\geq}$.

2. Why the tensor product? In classical probability, structural assessments of independence/dependence are expressed via expectations on factorised gambles $g(x_1, \ldots, x_m) = \prod_{j=1}^{m} g_j(x_j)$. This factorised gamble can equivalently be written as (1), see (Benavoli et al., 2019b) for more details.

3. Notice that, since g is a polynomial and Ω is bounded, $\min g = \inf g$ and $\max g = \sup g$.
4. The constant functions take the form $g(x_1, \ldots, x_m) = c(\otimes_{j=1}^{m} x_j)^\dagger I (\otimes_{j=1}^{m} x_j) = c$.
5. QM is a theory of bounded (algorithmic) rationality (Benavoli et al., 2019c,a). Generalised types of coherence were described in some detail in (Quaeghebeur et al., 2015).
6. The infimum coincides with the minimum because gambles are bounded polynomials.
7. In case of arbitrary set of assessments, we simply ask in addition for $\mathscr{C}$ to be topologically closed.
8. 'posi($\mathscr{A}$)' denotes the conic hull of a set of gambles $\mathscr{A}$. It is defined as $\text{posi}(\mathscr{A}) = \{\sum_i \lambda_i g_i : \lambda_i \in \mathbb{R}^{\geq}, g_i \in \mathscr{A}\}$.

Remark 1 *In classical coherence, the tautologies are the set of all nonnegative gambles $\mathscr{L}_R^{\geq}$. This is the only difference w.r.t. QM. The classical axioms of desirability are: (i) $\mathscr{L}_R^{\geq}$ should always be desirable; (ii) $\mathscr{K} := \text{posi}(\mathscr{L}_R^{\geq} \cup \mathscr{G})$; (iii) $-1 \notin \mathscr{K}$. However, evaluating if a gamble belongs to $\mathscr{L}_R^{\geq}$ is NP-hard as discussed previously.*

Remark 2 *There are different notions of desirability (almost, strict, real (Walley, 1991)); here we use the term desirability for almost desirability. A-coherence is an instance of almost desirability.*

We can finally associate a 'probabilistic' interpretation through the dual of an A-coherent set. Let us consider the dual space $\mathscr{L}_R^*$ of all bounded linear functionals $L : \mathscr{L}_R \to \mathbb{R}$. With the additional condition that linear functionals preserve the unitary gamble, the dual cone of an A-coherent $\mathscr{C} \subset \mathscr{L}_R$ is given by

$$\mathscr{C}^\circ := \{L \in \mathsf{S} \mid L(g) \geq 0,\ \forall g \in \mathscr{G}\}, \qquad (2)$$

where $\mathsf{S} = \{L \in \mathscr{L}_R^* \mid L(1) = 1,\ \ L(h) \geq 0\ \ \forall h \in \Sigma^{\geq}\}$ is the set of states. It is not difficult to prove that $\mathscr{C}^\circ$ can actually equivalently be defined as:

$$\mathscr{M} := \{\rho \in \mathcal{S} \mid Tr(G\rho) \geq 0,\ \ \forall g \in \mathscr{G}\}, \qquad (3)$$

where $\mathcal{S} = \{\rho \in \mathscr{H}^{n\times n} \mid \rho \geq 0,\ \ Tr(\rho) = 1\}$ is the set of all density matrices and gambles g are defined as in (1) and are essentially specified by the Hermitian matrix G. We also show that they are generalised moment[9] matrices: $\rho := L(zz^\dagger)$.

The derivation allows us to formulate quantum weirdness (that is the disagreement between QM and classical physics) as a Dutch book (sure loss). This goes as follows. Given that QM uses a stronger notion of positivity/negativity, a set of desirable gambles can include a gamble $f \in \mathscr{L}_R^{<} \backslash \Sigma^{<}$ and still be A-coherent. When this happens, we have entanglement. In this case, the experimental results appear illogical to us (incompatible with our common understanding), because they are simply incoherent under classical desirability.

9. In classical probability, given a (real) variable x and an expectation operator E, the n-th (non-central) moment of x is defined as $m_n := E[x^n]$ (we can also define multivariate moments, e.g., $E[x_1^n x_2^m]$). Given a sequence of moments $m_0, m_1, m_2, \ldots, m_n$, there exist infinitely many probability distributions corresponding to the same moments and they form a convex set. A sequence of scalars $m_0, m_1, m_2, \ldots, m_n$ is a valid sequence of moments provided that they satisfy certain consistency constraints. For instance, the moment matrix, obtained by organising that sequence into a matrix (in a certain way), must be positive semi-definite. This gives reason for the constraint $\rho \geq 0$ for density matrices in QT. In general, ρ is a generalised moment matrix, that is a moment matrix computed with respect to a 'charge'. (Benavoli et al., 2019b).

3. The Symmetrisation Postulate

In this section, we formulate the symmetrisation postulate using QM theory (Cohen-Tannoudji et al., 2020, XIV.C-1, p. 1434). In the next section, we will instead derive this postulate using exchangeable gambles.

Suppose we have m particles, each with single-particle state space represented by a vector space $V = \overline{\mathbb{C}}^n$ (we assume $n_j = n$, same dimension for all particles). We denote a state (a wavefunction) with $|\psi\rangle$, where $|\psi\rangle \in V$.[10] According to QM postulates, if the particles were distinguishable the composite space of m particles would be given by $\otimes_{i=1}^m V$. Let us denote the state of a particle with $|\alpha_i\rangle$, so that an element of $\otimes_{i=1}^m V$ is denoted as $|\psi\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle$.

Remark 3 *In section 2 we considered $x_i \in V$, while in this section we use $|\alpha_i\rangle \in V$. Why? The reason is that, in Section 2, x_i represents an unknown "classical" variable (e.g., the direction of the spin) and we ask a subject to express her beliefs about x_i in terms of acceptance of gambles. Conversely, $|\alpha_i\rangle$ is a state: a proxy quantity which is used in QM to compute the probability of the results of an experiment. QM postulates are formulated in terms of $|\alpha_i\rangle$ (usually denoted as $|\psi_i\rangle$). Indeed, under the epistemic interpretation of QM, $|\alpha_i\rangle$ corresponds to a belief state and so it is different from x_i. This difference is also evident from the fact that, for a composite system, $|\psi\rangle = |\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle \in \otimes_{i=1}^m V$, while $[x_1, \ldots, x_m] \in \times_{i=1}^m V$. To understand this difference, consider the toss of a classical coin: $\Omega = \{H, T\}$ and $p = [p_H, p_T] \in \mathbb{R}^2$ is the vector of probabilities for Heads and Tails. Now consider the toss of three coins, the composite possibility space is $\times_{i=1}^3 \Omega$, while the joint probability mass function belongs to $\otimes_{i=1}^3 \mathbb{R}^2 = \mathbb{R}^8$.*

In this work, we are interested in defining the state space for indistinguishable particles.

Let π denotes a permutation of the indices of the elements of the tensor product $|\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle$. Since such a permutation defines the product $|\alpha_{\pi(1)}\rangle \otimes \cdots \otimes |\alpha_{\pi(m)}\rangle$, by permuting the elements of the tensor products, we are basically permuting the labels of the particles. A permutation that only swaps two variables is called a *transposition*.

The *sign of a permutation* π, denoted by sign(π), equals 1 if π can be written as a product of an even number of transpositions, and equals -1 if π can be written as a product of an odd number of transpositions. Notice that the sign of π can be calculated as follows:

$$\text{sign}(\pi) = \det \sum_{i=1}^m e_i e_{\pi(i)}^T,$$

where e_i is an element of the canonical basis of $\mathbb{R}^m$ (see (Cohen-Tannoudji et al., 2020, XIV.B-2-c)).

Since permutations are linear operator, we can equivalently express permutation π as a matrix operator P_π acting

10. $|\psi\rangle$ is a ket, that is a column vector.

on the tensor product:

$$P_\pi(|\alpha_1\rangle \otimes \cdots \otimes |\alpha_m\rangle) := |\alpha_{\pi(1)}\rangle \otimes \cdots \otimes |\alpha_{\pi(m)}\rangle.$$

The matrix P_π is unitary, that is $P_\pi^\dagger P_\pi = P_\pi P_\pi^\dagger = I$, but not necessarily Hermitian. In what follows, by $\mathbb{P}_m$ we both denote the collection of all permutations and of all corresponding permutation operators.

We now introduce the *symmetriser* and the *antisymmetriser*:

$$\Pi_{\text{Sym}} := \frac{1}{m!}\sum_{\pi_r\in\mathbb{P}_m} P_{\pi_r},$$

$$\Pi_{\text{Anti}} := \frac{1}{m!}\sum_{\pi_r\in\mathbb{P}_m} \text{sign}(\pi_r)P_{\pi_r}.$$

which are projectors[11] (Cohen-Tannoudji et al., 2020, XIV.B-2-c). They project onto respectively:

$$\text{Sym}^m V = \{|\psi\rangle \in \otimes_{i=1}^m V : P_\pi|\psi\rangle = |\psi\rangle,\ \ \forall\pi\in\mathbb{P}_m\}$$
$$\text{Anti}^m V = \{|\psi\rangle \in \otimes_{i=1}^m V : P_\pi|\psi\rangle = \text{sign}(\pi)|\psi\rangle,\ \ \forall\pi\in\mathbb{P}_m\}.$$

Lemma 4 (Cohen-Tannoudji et al. (2020)) *The following equalities hold for any permutation operator $P_\pi \in \mathbb{P}_m$:*

1. $P_\pi\Pi_{\text{Sym}} = \Pi_{\text{Sym}}P_\pi = \Pi_{\text{Sym}}$;
2. $P_\pi\Pi_{\text{Anti}} = \Pi_{\text{Anti}}P_\pi = \text{sign}(\pi)\Pi_{\text{Anti}}$.

Proof Given two permutations $P_{\pi_i} \neq P_{\pi_j}$, we have that $P_\pi P_{\pi_i} \neq P_\pi P_{\pi_j}$. Hence we have that

$$P_\pi\Pi_{\text{Sym}} = \frac{1}{m!}\sum_{\pi_r\in\mathbb{P}_m} P_\pi P_{\pi_r} = \frac{1}{m!}\sum_{\pi'_r\in\mathbb{P}_m} P_{\pi'_r}.$$

Analogously, since $\text{sign}(\pi)\text{sign}(\pi) = 1$

$$\begin{aligned} P_\pi\Pi_{\text{Anti}} &= \frac{1}{m!}\sum_{\pi_r\in\mathbb{P}_m} \text{sign}(\pi_r)P_\pi P_{\pi_r} \\ &= \frac{\text{sign}(\pi)}{m!}\sum_{\pi_r\in\mathbb{P}_m} \text{sign}(\pi_r)\text{sign}(\pi)P_\pi P_{\pi_r} \\ &= \frac{\text{sign}(\pi)}{m!}\sum_{\pi'_r\in\mathbb{P}_m} \text{sign}(\pi'_r)P_{\pi'_r}. \end{aligned}$$

■

The *symmetrisation postulate* states the following:

> When a system includes several identical particles, only certain states of its state space can describe its physical states. Physical states are, depending on the nature of the identical particles, either completely symmetric or completely antisymmetric with respect to permutation of these particles. Those particles for which the physical states are symmetric are called bosons, and those for which they are antisymmetric, fermions. (Cohen-Tannoudji et al., 2020, XIV.C-1, p. 1434)

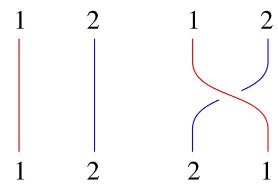
The postulate thus limits the state space (possibility space) for a system of identical particles. Contrary to the case of particles of different natures, this space is no longer the tensor product $\otimes_{i=1}^m V$ of the individual state spaces of the particles constituting the system, but rather a subspace, namely $\text{Sym}^m V$ or $\text{Anti}^m V$, depending on whether the particles are bosons or fermions. Only states belonging either to $\text{Sym}^m V$ or to $\text{Anti}^m V$ are physically possible. This is the reasons they are called *physical states*.

Given k physical states $|\psi_i\rangle$ (belonging to either $\text{Sym}^m V$ or $\text{Anti}^m V$), we can then define the density matrix as usual:

$$\rho = \sum_{i=1}^k p_i|\psi_i\rangle\langle\psi_i|,$$

where p_i are probabilities, $p_i \geq 0$ and $\sum_{i=1}^k p_i = 1$. It can then be verified that, in the symmetric case, given that $|\psi_i\rangle = \Pi_{\text{Sym}}|\psi_i\rangle$, we have that $\rho = \Pi_{\text{Sym}}\rho\Pi_{\text{Sym}}$. Similarly, in the antisymmetric case, $\rho = \Pi_{\text{Anti}}\rho\Pi_{\text{Anti}}$.

Example 1 *Consider $m = 2$ particles with $|\alpha_1\rangle, |\alpha_2\rangle \in \overline{\mathbb{C}}^2$. In this case there are only two possible permutations π_a (identity) and π_b (swap) with $\text{sign}(\pi_b) = -1$:*



The permutation matrices are $P_{\pi_a} = I$ and:

$$P_{\pi_b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

The latter acts on $|\alpha_1\rangle \otimes |\alpha_2\rangle$ as follows

$$P_{\pi_b}(|\alpha_1\rangle\otimes|\alpha_2\rangle) = P_{\pi_b}\begin{bmatrix} \alpha_{11}\alpha_{21} \\ \alpha_{11}\alpha_{22} \\ \alpha_{12}\alpha_{21} \\ \alpha_{12}\alpha_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}\alpha_{21} \\ \alpha_{12}\alpha_{21} \\ \alpha_{11}\alpha_{22} \\ \alpha_{12}\alpha_{22} \end{bmatrix} = |\alpha_2\rangle\otimes|\alpha_1\rangle.$$

The projectors are:

$$\Pi_{\text{Sym}} = \frac{I + P_{\pi_b}}{2},\ \ \Pi_{\text{Anti}} = \frac{I + sign(\pi_b)P_{\pi_b}}{2} = \frac{I - P_{\pi_b}}{2}, \tag{5}$$

11. They are Hermitian $\Pi_{\text{Sym}}^\dagger = \Pi_{\text{Sym}}, \Pi_{\text{Anti}}^\dagger = \Pi_{\text{Anti}}$ and they satisfy $\Pi_{\text{Sym}}^2 = \Pi_{\text{Sym}}, \Pi_{\text{Anti}}^2 = \Pi_{\text{Anti}}$ and $\Pi_{\text{Sym}}\Pi_{\text{Anti}} = \Pi_{\text{Anti}}\Pi_{\text{Sym}} = 0$.

which act on $|\alpha_1\rangle \otimes |\alpha_2\rangle$ *as follows*[12]

$$\Pi_{\text{Sym}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = \begin{bmatrix} \alpha_{11}\alpha_{21} \\ \frac{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}}{2} \\ \frac{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}}{2} \\ \alpha_{12}\alpha_{22} \end{bmatrix}, \tag{6}$$

$$\Pi_{\text{Anti}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = \begin{bmatrix} 0 \\ \frac{\alpha_{11}\alpha_{22}-\alpha_{12}\alpha_{21}}{2} \\ \frac{\alpha_{12}\alpha_{21}-\alpha_{11}\alpha_{22}}{2} \\ 0 \end{bmatrix} \tag{7}$$

From last equality, it follows that, in case $\alpha_1 = \alpha_2$, $\Pi_{\text{Anti}}(|\alpha_1\rangle \otimes |\alpha_2\rangle) = 0$*. This is called* Pauli exclusion principle*: two Fermions cannot have identical state.*

4. Exchangeable Gambles

In the previous section, we discussed the symmetrisation postulate. In this section, we formulate it in terms of A-coherence and exchangeability. In doing so, we extend some of the definitions and results originally presented in (De Cooman and Quaeghebeur, 2012) to the quantum setting introduced in Section 2.

As discussed in Section 2, we consider gambles on $x_i \in V = \overline{\mathbb{C}}^n$. Given m particles, the possibility space is $\times_{i=1}^m V$. Therefore, π denotes a permutation of the indices of the vector $(x_1,\ldots,x_m)$, i.e.,

$$\pi(x_1,\ldots,x_m) = (x_{\pi(1)},\ldots,x_{\pi(m)}).$$

A generic gamble is denoted as:

$$g(z,z) := z^\dagger G z,$$

with $z := \otimes_{j=1}^m x_j$. Let π_r, π_l be two permutations, we define

$$\begin{aligned}\pi_l g(z,z)\pi_r &:= \frac{1}{2}\left(g(\pi_l z, \pi_r z) + g(\pi_r z, \pi_l z)\right) \\ &= \frac{1}{2}\left(z^\dagger P_{\pi_l}^\dagger G P_{\pi_r} z + z^\dagger P_{\pi_r}^\dagger G P_{\pi_l} z\right).\end{aligned}$$

Note that (i) $\pi_l g \pi_r = \pi_r g \pi_l$, and (ii) $\pi_l g \pi_r$ is a gamble (it returns real values).[13]

Remark 5 *This definition of permuted gamble is different from the one used in (De Cooman and Quaeghebeur, 2012) (the permutation of* $g(\omega)$ *is defined as* $\pi \circ g = g(\pi\omega)$*). In QM, gambles are quadratic forms of complex variables and, therefore, we can define more general symmetries by exploiting the fact that* z *and its complex conjugate* $z^\dagger$ *can be treated as two "different" variables.*

12. The right hand side term in (6) or (7) is a complex vector, but its norm can be different from one. In this latter case, it needs to be normalised.

13. This holds because $P_{\pi_l}^\dagger G P_{\pi_r} + P_{\pi_r}^\dagger G P_{\pi_l}$ is Hermitian.

Example 2 *Consider* $m = 2$ *particles with* $x_1, x_2 \in \overline{\mathbb{C}}^2$*. We have already seen that there are only two possible permutations* π_a *(identity) and* π_b *(swap). Therefore, we have* $\pi_a g \pi_a = g$ *and*

$$\pi_a g \pi_b = \frac{1}{2}\left((x_1 \otimes x_2) G (x_2 \otimes x_1) + (x_2 \otimes x_1) G (x_1 \otimes x_2)\right),$$
$$\pi_b g \pi_b = (x_2 \otimes x_1) G (x_2 \otimes x_1).$$

For $\pi_l, \pi_r \in \mathbb{P}_m$, we write

$$\delta_{l,r}^\star := \begin{cases} \text{sign}(\pi_l)\text{sign}(\pi_r) & \text{when } \star = Anti, \\ 1 & \text{when } \star = Sym. \end{cases}$$

Given this definition, in the remaining of this section, all definitions, results and corresponding proofs will be parameterised by $\star \in \{Anti, Sym\}$ and $\delta_{l,r}^\star$. They therefore apply, uniformly, to both the symmetric and the antisymmetric cases.

We now provide the definition of A-coherent $\star$-exchangeable set of desirable gambles.

Definition 6 *Consider the set*

$$\mathscr{I}_\star := \{g - \delta_{l,r}^\star \pi_l g \pi_r \mid g \in \mathscr{L}_R,\ \pi_l, \pi_r \in \mathbb{P}_m\}.$$

We say that an A-coherent set of desirable gambles $\mathscr{C}$ *is* $\star$*-exchangeable if* $\mathscr{I}_\star \subseteq \mathscr{C}$*.*

Given Definition 6, we can prove the following result.

Proposition 7 *Let* $\mathscr{C}$ *be an A-coherent set of desirable gambles. If* $\mathscr{C}$ *is* $\star$*-exchangeable, then it is also* $\star$*-permutable, that is* $\delta_{l,r}^\star \pi_l g \pi_r$ *are in* $\mathscr{C}$ *for all* $g \in \mathscr{C}$ *and all* $\pi_l, \pi_r \in \mathbb{P}_m$*.*

Proof The proof is similar as the one for (De Cooman and Quaeghebeur, 2012, Prop.9). For $g \in \mathscr{C}$ and $\pi_l, \pi_r \in \mathbb{P}_m$, we have $-g - \delta_{l,r}^\star \pi_l(-g)\pi_r \in \mathscr{I}_\star \subseteq \mathscr{C}$. Given that $-g = z^\dagger(-G)z$, then $-g - \delta_{l,r}^\star \pi_l(-g)\pi_r = \delta_{l,r}^\star \pi_l g \pi_r - g$. Since $\delta_{l,r}^\star \pi_l g \pi_r = \delta_{l,r}^\star \pi_l g \pi_r - g + g$ and $g, \delta_{l,r}^\star \pi_l g \pi_r - g \in \mathscr{C}$, we conclude by additivity that $\delta_{l,r}^\star \pi_l g \pi_r \in \mathscr{C}$. ■

As in (De Cooman and Quaeghebeur, 2012), but taking into account that we are working with quadratic forms, we define the linear operators

$$\text{ex}_\star^m(g) := z^\dagger \Pi_\star^\dagger G \Pi_\star z.$$

We verify some of their properties; in particular that they can be used to equivalently characterise symmetric and antisymmetric exchangeability (Corollary 11).

The first result follows immediately from the fact that the symmetrisers and the antisymmetriser are projectors.

Lemma 8 *Let* g *be a gamble, then* $ex_\star^m(ex_\star^m(g)) = ex_\star^m(g)$*.*

The idea behind this linear transformations $\mathrm{ex}_\star^m(g)$ is that they render a gamble g insensitive to permutation by replacing it with the uniform average $\mathrm{ex}_\star^m(g)$ of all its permutations $\pi_l g \pi_r$, as shown hereafter.

Proposition 9 *Let g be a gamble, then*

$$ex_\star^m(g) = \frac{1}{m!m!} \sum_{\pi_r,\pi_l \in \mathbb{P}_m} \delta^\star_{l,r} \pi_l g \pi_r.$$

Proof It is immediate to verify that $\mathrm{ex}_\star^m(g) = \frac{1}{m!m!}\sum_{\pi_r,\pi_l\in\mathbb{P}_m} \delta^\star_{l,r} g(\pi_l z, \pi_r z)$. To conclude, note that:

$$\begin{aligned}
&\sum_{\pi_l,\pi_r\in\mathbb{P}_m} \delta^\star_{l,r}\pi_l g \pi_r = \\
&= \sum_{\pi_l,\pi_r\in\mathbb{P}_m} \frac{\delta_\star}{2}\left(g(\pi_l z,\pi_r z)+g(\pi_r z,\pi_l z)\right)\\
&= \frac{1}{2}\sum_{\pi_l,\pi_r\in\mathbb{P}_m} \delta^\star_{l,r} g(\pi_l z,\pi_r z) + \frac{1}{2}\sum_{\pi_l,\pi_r\in\mathbb{P}_m} \delta^\star_{r,l} g(\pi_r z,\pi_l z)\\
&= \sum_{\pi_l,\pi_r\in\mathbb{P}_m} \delta^\star_{l,r} g(\pi_l z,\pi_r z).
\end{aligned}$$

■

Clearly, the linear transformations $\mathrm{ex}_\star^m$ assume the same value for all gambles that can be related to each other through some permutation.

Proposition 10 *Let g be a gamble, and $\pi_l,\pi_r \in \mathbb{P}_m$. Then*

$$ex_\star^m(\delta^\star_{l,r}\pi_l g \pi_r) = ex_\star^m(g).$$

Proof By exploiting linearity

$$\begin{aligned}
&\mathrm{ex}_\star^m(\delta^\star_{l,r}\pi_l g\pi_r) = \delta^\star_{l,r}\mathrm{ex}_\star^m(\pi_l g \pi_r) = \\
&= \delta^\star_{l,r}\left(z^\dagger \Pi_\star^\dagger \left(\frac{1}{2}\left(P^\dagger_{\pi_l} G P_{\pi_r} + P^\dagger_{\pi_r} G P_{\pi_l}\right)\right)\Pi_\star z\right)\\
&= \frac{\delta^\star_{l,r}}{2} z^\dagger \Pi_\star^\dagger P^\dagger_{\pi_l} G P_{\pi_r}\Pi_\star z + \frac{\delta^\star_{l,r}}{2} z^\dagger \Pi_\star^\dagger P^\dagger_{\pi_r} G P_{\pi_l}\Pi_\star z\\
&= \frac{\delta^\star_{l,r}}{2} z^\dagger (P_{\pi_l}\Pi_\star)^\dagger G (P_{\pi_r}\Pi_\star) z + \frac{\delta^\star_{l,r}}{2} z^\dagger (P_{\pi_r}\Pi_\star)^\dagger G (P_{\pi_l}\Pi_\star) z
\end{aligned}$$

By Lemma 4 and the fact that $\delta^\star_{l,r}\delta^\star_{l,r}=1$, we finally obtain

$$\begin{aligned}
&\frac{\delta^\star_{l,r}}{2} z^\dagger (P_{\pi_l}\Pi_\star)^\dagger G (P_{\pi_r}\Pi_\star) z + \frac{\delta^\star_{l,r}}{2} z^\dagger (P_{\pi_r}\Pi_\star)^\dagger G (P_{\pi_l}\Pi_\star) z = \\
&= \frac{\delta^\star_{l,r}\delta^\star_{l,r}}{2} z^\dagger \Pi_\star^\dagger G \Pi_\star z + \frac{\delta^\star_{l,r}\delta^\star_{l,r}}{2} z^\dagger \Pi_\star^\dagger G \Pi_\star z\\
&= \mathrm{ex}_\star^m(g).
\end{aligned}$$

■

Similarly to what was done by De Cooman and Quaeghebeur (2012), we can prove the following.

Corollary 11 *Let $\mathscr{C}$ be an A-coherent set of desirable gambles. Given*

$$\mathscr{V}_\star := \{g - ex_\star^m(g) \mid g \in \mathscr{L}_R\}$$

the following claims are equivalent,

(1) *$\mathscr{C}$ is $\star$-exchangeable;*

(2) *$\mathscr{V}_\star \subseteq \mathscr{C}$.*

Proof For ($1 \Rightarrow 2$), by Proposition 9, we can write $g - \mathrm{ex}_\star^m(g) = \frac{1}{m!m!}\sum_{\pi_l\pi_r}(g - \delta^\star_{l,r}\pi_l g \pi_r)$. Since $\mathscr{C}$ satisfies linearity and given $\mathscr{I}_\star \subseteq \mathscr{C}$, then $g - \mathrm{ex}_\star^m(g) \in \mathscr{C}$.

For ($2 \Rightarrow 1$), by linearity of $\mathrm{ex}_\star^m$ and Proposition 10

$$g - \delta^\star_{l,r}\pi_l g \pi_r - \mathrm{ex}_\star^m(g - \delta^\star_{l,r}\pi_l g\pi_r) = g - \delta^\star_{l,r}\pi_l g \pi_r,$$

which shows that $g - \delta^\star_{l,r}\pi_l g\pi_r \in \mathscr{C}$. ■

The following result also holds.

Proposition 12 *Let $\mathscr{C}$ be an A-coherent set of desirable gambles. Then, assuming $\mathscr{C}$ is $\star$-exchangeable, the following claims hold for all gambles g, g':*

1. *$g \in \mathscr{C}$ iff $ex_\star^m(g) \in \mathscr{C}$;*
2. *if $ex_\star^m(g) = ex_\star^m(g')$ then $g \in \mathscr{C}$ iff $g' \in \mathscr{C}$.*

Proof The proof is the same as for (De Cooman and Quaeghebeur, 2012, Prop.10). First notice that the first claim follows from the second, by taking $g' := \mathrm{ex}_\star^m(g)$ and applying Lemma 8. For the second claim, assume $\mathrm{ex}_\star^m(g) = \mathrm{ex}_\star^m(g')$ and $g \in \mathscr{C}$. Notice that $g' - \mathrm{ex}_\star^m(g') = g' - \mathrm{ex}_\star^m(g), -g - \mathrm{ex}_\star^m(-g) = \mathrm{ex}_\star^m(g) - g \in \mathscr{V}_\star$. By Corollary 11 and additivity, we obtain $(g' - \mathrm{ex}_\star^m(g)) + (\mathrm{ex}_\star^m(g) - g) + g = g' \in \mathscr{C}$. ■

We now consider the dual of an A-coherent set of $\star$-exchangeable gambles.

From Section 2, to define the dual, we consider the dual space $\mathscr{L}_R^*$ of all bounded linear functionals $L : \mathscr{L}_R \to \mathbb{R}$. With the additional condition that linear functionals preserve the unitary gamble, the dual cone of an A-coherent $\mathscr{C} \subset \mathscr{L}_R$ is given by

$$\mathscr{C}^\circ = \{L \in \mathsf{S} \mid L(g) \geq 0,\ \forall g \in \mathscr{G}\}, \tag{8}$$

where $\mathsf{S} = \{L \in \mathscr{L}_R^* \mid L(1) = 1,\ \ L(h) \geq 0\ \ \forall h \in \Sigma^{\geq}\}$ is the set of states.

Definition 13 *Let $L \in \mathsf{S}$. We say that L is* $\star$-exchangeable *if it belongs to the dual $\mathscr{C}^\circ$ of an A-coherent $\star$-exchangeable set of gambles $\mathscr{C}$.*

Proposition 14 *Assume $L \in \mathsf{S}$. The following statements are equivalent:*

1. L is $\star$-exchangeable;
2. $L(f) = 0$ for all $f \in \mathscr{I}_\star$.
3. $L(f) = 0$ for all $f \in \mathscr{V}_\star$.

Proof We verify (1⇔2). If L is $\star$-exchangeable, we know that $g - \delta^\star_{l,r}\pi_l g \pi_r, \delta^\star_{l,r}\pi_l g \pi_r - g \in \mathscr{C}$, meaning that $L(g - \delta^\star_{l,r}\pi_l g\pi_r) \geq 0$ and $-L(g - \delta^\star_{l,r}\pi_l g\pi_r) \geq 0$. Therefore $L(f) = L(g - \delta^\star_{l,r}\pi_l g\pi_r) = 0$. For the other direction, assume that $L(f) = 0$ for all $f \in \mathscr{I}_\star$, From L, by duality, we can define the set of desirable gambles $\{g \in \mathscr{L}_R : L(g) \geq 0\}$. We have proven in (Benavoli et al., 2019b) that this is an A-coherent set of desirable gamble and, moreover, it includes $\mathscr{I}_\star$ by hypothesis. By Corollary 11, the equivalence (1⇔3) can be proven in a similar way. ■

We recall the following well-know result (see e.g. (Holevo, 2011)).

Proposition 15 *Let G be a Hermitian matrix; then $G \geq 0$ if and only if $Tr(SG) \geq 0$ for all $S \geq 0$.*

We use the previous result to prove the following.

Proposition 16 *Assume $L \in \mathsf{S}$. The following statements are equivalent:*

1. L is $\star$-exchangeable;
2. $L\left(zz^\dagger - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}zz^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}zz^\dagger P^\dagger_{\pi_r}\right) = 0$ for all $\pi_l\pi_r \in \mathbb{P}_m$;
3. $L\left(zz^\dagger - \Pi_\star zz^\dagger \Pi^\dagger_\star\right) = 0$.

Proof As before, we only verify the equivalence (1⇔2). Assume $L \in \mathsf{S}$ is $\star$-exchangeable and consider the set of gambles $\mathscr{A} = \{g - \delta^\star_{l,r}\pi_l g\pi_r : \pi_l\pi_r \in \mathbb{P}_m, G \geq 0\}$ and $\mathscr{B} = \{\delta^\star_{l,r}\pi_l g\pi_r - g : \pi_l\pi_r \in \mathbb{P}_m,\ G \geq 0\}$. Since L is $\star$-exchangeable, it follows that $L(f), L(f') \geq 0$ for each $f \in \mathscr{A}, f' \in \mathscr{B}$. This implies that

$$
\begin{aligned}
0 &\leq L(g - \delta^\star_{l,r}\pi_l g\pi_r) \\
&= L(z^\dagger G z) - \frac{\delta^\star_{l,r}}{2} L\left(z^\dagger P^\dagger_{\pi_l} G P_{\pi_r} z + z^\dagger P^\dagger_{\pi_r} G P_{\pi_l} z\right) \\
&= Tr\left(GL\left(zz^\dagger - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}zz^\dagger P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}zz^\dagger P^\dagger_{\pi_r}\right)\right) \\
&= Tr\left(G\left(L(zz^\dagger) - \frac{\delta^\star_{l,r}}{2}P_{\pi_r}L(zz^\dagger)P^\dagger_{\pi_l} - \frac{\delta^\star_{l,r}}{2}P_{\pi_l}L(zz^\dagger)P^\dagger_{\pi_r}\right)\right)
\end{aligned}
$$

for each $\pi_l\pi_r \in \mathbb{P}_m,\ G \geq 0$. Similarly, we have that $0 \leq L(-g + \delta^\star_{l,r}\pi_l g\pi_r) = -L(g - \delta^\star_{l,r}\pi_l g\pi_r)$. We therefore conclude the proof of this implication by applying Proposition 15. To prove the other direction, simply note that the second claim implies that $0 = L(-g + \delta^\star_{l,r}\pi_l g\pi_r) = -L(g - \delta^\star_{l,r}\pi_l g\pi_r)$. ■

From (Benavoli et al., 2019b), we know that $\rho := L(zz^\dagger)$ is indeed a density matrix. Therefore, Proposition 16 immediately implies the following.

Corollary 17 *A density matrix $\rho \in \mathbb{S} = \{\rho \in \mathscr{H}^{n\times n} \mid \rho \geq 0,\ Tr(\rho) = 1\}$ is $\star$- exchangeable if any of the following statement holds:*

1. $\rho = \frac{\delta^\star_{l,r}}{2}P_{\pi_r}\rho P^\dagger_{\pi_l} + \frac{\delta^\star_{l,r}}{2}P_{\pi_l}\rho P^\dagger_{\pi_r}$ *for all* $\pi_l\pi_r \in \mathbb{P}$;
2. $\rho = \Pi_\star \rho \Pi^\dagger_\star$.

Point 2 of Corollary 17 therefore derives the *symmetrisation postulate* discussed in Section 3 via duality from a set of A-coherent exchangeable gambles.

Example 3 *Consider the density matrix*

$$
\begin{aligned}
\rho := L&\left(\begin{bmatrix} x_{11}x^\dagger_{11}x_{21}x^\dagger_{21} & x^\dagger_{11}x_{12}x_{21}x^\dagger_{21} & x_{11}x^\dagger_{11}x^\dagger_{21}x_{22} & x^\dagger_{11}x_{12}x^\dagger_{21}x_{22} \\ x_{11}x^\dagger_{12}x_{21}x^\dagger_{21} & x_{12}x^\dagger_{12}x_{21}x^\dagger_{21} & x_{11}x^\dagger_{12}x^\dagger_{21}x_{22} & x_{12}x^\dagger_{12}x^\dagger_{21}x_{22} \\ x_{11}x^\dagger_{11}x_{21}x^\dagger_{22} & x^\dagger_{11}x_{12}x_{21}x^\dagger_{22} & x_{11}x^\dagger_{11}x_{22}x^\dagger_{22} & x^\dagger_{11}x_{12}x_{22}x^\dagger_{22} \\ x_{11}x^\dagger_{12}x_{21}x^\dagger_{22} & x_{12}x^\dagger_{12}x_{21}x^\dagger_{22} & x_{11}x^\dagger_{12}x_{22}x^\dagger_{22} & x_{12}x^\dagger_{12}x_{22}x^\dagger_{22} \end{bmatrix}\right) \\
&= \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{9}
$$

For $P_{\pi_a} = I_4$ and P_{π_b} as in (4), we have

$$\rho = P^\dagger_{\pi_a}\rho P_{\pi_b} = P^\dagger_{\pi_b}\rho P_{\pi_a} = P^\dagger_{\pi_b}\rho P_{\pi_b}.$$

Therefore, ρ is symmetrically exchangeable (it also satisfies $\Pi_{\text{Sym}}\rho\Pi_{\text{Sym}} = \rho$.) Instead the matrix

$$\rho = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

is antisymmetrically exchangeable. It satisfies $\Pi_{\text{Anti}}\rho\Pi_{\text{Anti}} = \rho$ as well as $\rho = -0.5(P^\dagger_{\pi_a}\rho P_{\pi_b} + P^\dagger_{\pi_b}\rho P_{\pi_a}) = P^\dagger_{\pi_b}\rho P_{\pi_b}$.

5. Updating

Let us assume we measure a subset of particles $x_1, \ldots, x_{\breve{m}}$ with $\breve{m} \leq m$. Quantum projection measurements can then be described by a collection $\{\Pi_i\}_{i=1}^{n\breve{m}}$, with $\Pi_i \in \mathscr{H}^{n\breve{m}\times n\breve{m}}$, of projection operators that satisfy the completeness equation $\sum_{i=1}^{n\breve{m}} \Pi_i = I$.

We recall the following definition from (Benavoli et al., 2019b, Sec. S.9.1).

Definition 18 *Let $\mathscr{C}$ be an A-coherent $\star$-exchangeable coherent set of desirable gambles, the set obtained as*

$$\mathscr{C}_{\Pi_i} = \left\{z^\dagger G z \mid z^\dagger(\Pi_i \otimes I_{m-\breve{m}})^\dagger G(\Pi_i \otimes I_{m-\breve{m}})z \in \mathscr{C}\right\} \tag{11}$$

is called the **set of desirable gambles conditional** *on Π_i.*

We already know (Benavoli et al., 2016) that updating preserves coherence. We now see that it also preserves exchangeability.

Proposition 19 *Let $\mathscr{C}$ be an A-coherent $\star$-exchangeable coherent set of desirable gambles. Then $\mathscr{C}_{\Pi_i}$ is an A-coherent $\star$-exchangeable coherent set of desirable gambles on the variables $x_{\check{m}+1},\dots,x_m$ and its dual is*

$$\mathscr{M}_{\Pi_i}=\left\{\frac{(\Pi_i\otimes I_{m-\check{m}})^\dagger\rho(\Pi_i\otimes I_{m-\check{m}})}{Tr((\Pi_i\otimes I_{m-\check{m}})^\dagger\rho(\Pi_i\otimes I_{m-\check{m}}))}\middle|\rho\in\mathscr{M}\right\}, \tag{12}$$

where $\mathscr{M}$ is the dual of $\mathscr{C}$.

Proof In (Benavoli et al., 2019b, Sec. S.9.1) we have already proved that $\mathscr{C}_{\Pi_i}$ is coherent and that $\mathscr{M}_{\Pi_i}$ is the dual of $\mathscr{C}_{\Pi_i}$. Therefore, we only need to prove that $\mathscr{C}_{\Pi_i}$ is a $\star$-exchangeable coherent set of desirable gambles on the variables $x_{\check{m}+1},\dots,x_m$. This means we need to prove that $z^\dagger Gz-\left(\frac{\delta^\star_{l,r}}{2}z^\dagger(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})^\dagger G(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})z+\frac{\delta^\star_{l,r}}{2}z^\dagger(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})^\dagger G(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})z\right)\in\mathscr{C}_{\Pi_i}$ for each gamble $z^\dagger Gz$. This gamble is in $\mathscr{C}_{\Pi_i}$ provided that:

$$z^\dagger(\Pi_i\otimes I_{m-\check{m}})^\dagger\Big[G-\frac{\delta^\star_{l,r}}{2}(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})^\dagger G(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})$$
$$-\frac{\delta^\star_{l,r}}{2}(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})^\dagger G(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})\Big](\Pi_i\otimes I_{m-\check{m}})z,$$

is in $\mathscr{C}$. By exploiting the following property of the tensor product

$$(I_2\otimes B)(A\otimes I_1)=(A\otimes I_1)(I_2\otimes B),$$

we need to verify that

$$z^\dagger\Big[(\Pi_i\otimes I_{m-\check{m}})^\dagger G(\Pi_i\otimes I_{m-\check{m}})$$
$$-\frac{\delta^\star_{l,r}}{2}(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})^\dagger(\Pi_i\otimes I_{m-\check{m}})^\dagger G(\Pi_i\otimes I_{m-\check{m}})(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})$$
$$-\frac{\delta^\star_{l,r}}{2}(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_r})^\dagger(\Pi_i\otimes I_{m-\check{m}})^\dagger G(\Pi_i\otimes I_{m-\check{m}})(I_{\check{m}}\otimes P^{m-\check{m}}_{\pi_l})\Big]z,$$

is in $\mathscr{C}$. This is true because $\mathscr{C}$ is $\star$-exchangeable. ■

6. Entanglement

Unlike systems consisting of distinguishable[14] particles, in the case of identical particles the notion of entanglement is still under debate (see e.g. (Benatti et al., 2014)). The reason being that, for instance, the two matrices in Example 3 are entangled density matrices for distinguishable particles and, at the same time, they also satisfy the symmetry and anti-symmetry relationship of identical particles. Are those density matrices entangled in the (anti-)symmetric case?

For distinguishable particles, our gambling formulation of QM allows us to formulate and detect entangled density matrices thorough a Dutch book (sure loss) (Benavoli et al., 2019b). This goes as follows. Given a density matrix $\tilde{\rho}$, we can first compute its dual (an A-coherent set of desirable gambles):

$$\mathscr{C}:=\{g(z,z)=z^\dagger Gz:L(g)=Tr(G\tilde{\rho})\geq 0\}$$

and then consider its "classical" extension

$$\mathscr{K}:=\text{posi}(\mathscr{C}\cup\mathscr{L}_R^{\geq}).$$

Hence, $\mathscr{K}$ is coherent (under the standard axioms of desirability) provided that $\mathscr{K}\cap\mathscr{L}_R^{<}=\emptyset$.

As done in (Benavoli et al., 2019b, Sec.4.4), we thus state the following definition.

Definition 20 *Let $\tilde{\rho}$ be a density matrix. Then $\tilde{\rho}$ is entangled if $\mathscr{K}\cap\mathscr{L}_R^{<}\neq\emptyset$ ($\mathscr{K}$ does not avoid sure loss).*

If $\tilde{\rho}$ is not entangled, the bounded linear functionals in its dual can be written as an integral with respect to a probability measure (Benavoli et al., 2019b):[15]

$$\rho=\int_\Omega zz^\dagger d\mu(z). \tag{13}$$

As an immediate consequence of Definition 20 and Equation (13) we get:

Proposition 21 *Let $\tilde{\rho}$ be a density matrix, then $\tilde{\rho}$ is not entangled iff it is a truncated moment matrix (with respect to a standard probability measure $\mu(z)$).*

The question is therefore how we can extend this result to the case of indistinguishable particles. In this aim, we need to consider a constraint: *not all Dutch books can be constructed*. In a system of indistinguishable particles, *physical observables* (that is, gambles which can be evaluated through an experiment or, equivalently, physically realisable gambles) must be invariant under all permutations of the m identical particles (Cohen-Tannoudji et al., 2020, XIV.C-4-a):

$$g(z,z)=z^\dagger Gz=z^\dagger\Pi_\star G\Pi_\star z\ \ \forall z. \tag{14}$$

Based on this constraint, we can thus obtain the following result.

Proposition 22 *Let $\tilde{\rho}$ be an entangled $\star$-exchangeable density matrix, then the following two statements are equivalent:*

- *there exists a physical observable $g(z,z)$ which belongs to $\mathscr{L}_R^{<}$ such that $Tr(G\tilde{\rho})\geq 0$;*

14. Spatially well-separated indistinguishable particles can be distinguished.

15. To do that, we need to perform another natural extension to the space of all gambles posi$(\mathscr{K}\cup\mathscr{L}^{\geq})$.

- $\tilde{\rho}$ *cannot be written as*

$$\int_{\Omega} \frac{\Pi_{\star} z z^{\dagger} \Pi_{\star}^{\dagger}}{Tr(\Pi_{\star} z z^{\dagger} \Pi_{\star}^{\dagger})} d\mu(z).$$

for any probability measure $\mu(z)$.

Proof The results follow by (Benavoli et al., 2019b, Sec.4.4). We only need to observe that if $g(z,z) < 0$, then for all probability measures μ:

$$\begin{aligned} 0 &> \int_{\Omega} z^{\dagger} G z d\mu(z) \\ &= \int_{\Omega} z^{\dagger} \Pi_{\star} G \Pi_{\star} z d\mu(z) \\ &= \int_{\Omega} Tr(G \Pi_{\star} z z^{\dagger} \Pi_{\star}) d\mu(z) \\ &= Tr\left(G \int_{\Omega} \Pi_{\star} z z^{\dagger} \Pi_{\star} d\mu(z) \right), \end{aligned}$$

the second equality follows by the assumption that g is a *physical observable* and thus Equation (14). The above inequality implies that

$$Tr\left(G \int_{\Omega} \frac{\Pi_{\star} z z^{\dagger} \Pi_{\star}^{\dagger}}{Tr(\Pi_{\star} z z^{\dagger} \Pi_{\star}^{\dagger})} d\mu(z) \right) < 0.$$

Let σ_{μ} be the result of the integral (a density matrix). Since the above inequality must hold for all μ, we can rewrite this condition as $\sup_{\mu} Tr(G\sigma_{\mu}) < 0$. Therefore, any density matrix $\tilde{\rho}$ such that $Tr(G\tilde{\rho}) \geq 0$ must be entangled: it cannot be expressed as an expectation with respect to a probability measure μ. ■

The above result means that particles are entangled when classical probability coherence and quantum A-coherence disagree. The first statement tells us that we can use a Dutch book (sure loss) to detect entanglement, but only if the Dutch book is a physical observable. Notice that Proposition 22 is in agreement with definitions of entanglement, and ways to detect it, as discussed for instance in (Iemini et al., 2013; Iemini and Vianna, 2013; Reusch et al., 2015) (in particular see (Reusch et al., 2015, Eq. 12)).

Example 4 *We apply Proposition 22 to the previous two particles Example 3.*

Fermions: consider the atomic charge (Dirac's delta) $\mu = \delta_{\tilde{z}}$ *with* $\tilde{z} = [1,0]^T \otimes [0,1]^T = [0,1,0,0]^T$. *Note that,*

$$\int_{\Omega} \frac{\Pi_{\text{Anti}} z z^{\dagger} \Pi_{\text{Anti}}^{\dagger}}{Tr(\Pi_{\text{Anti}} z z^{\dagger} \Pi_{\text{Anti}}^{\dagger})} \delta_{\tilde{z}}(z). = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the density matrix in (10). From Proposition 22, we conclude that that the density matrix is not entangled.

Bosons: consider the atomic charge (Dirac's delta) $\delta_{\tilde{z}}$ *with* $\tilde{z} = \frac{1}{2}[-\iota,1]^T \otimes [\iota,1]^T = [0.5,-0.5\iota,0.5\iota,0.5]^T$, *where* ι *is the complex unit. Note that,*

$$\int_{\Omega} \frac{\Pi_{\text{Sym}} z z^{\dagger} \Pi_{\text{Sym}}^{\dagger}}{Tr(\Pi_{\text{Sym}} z z^{\dagger} \Pi_{\text{Sym}}^{\dagger})} \delta_{\tilde{z}}(z). = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

which is the density matrix in (9). From Proposition 22, we conclude that that the density matrix is not entangled.

7. Conclusions

In this paper we showed that we can derive the *symmetrization postulate* for indistinguishable particles in QM using the framework of exchangeable desirable gambles. Therefore, once again, we proved that QM is a theory of probability: it can be derived by the principles of coherence and computation plus structural assessments of exchangeability. Moreover, we showed that, also in the case of indistinguishable particles, entanglement can be defined (and detected) as a Dutch book: the clash between the QM notion of rationality (which accounts for the principle of computation) and the classical notion of rationality (which assumes infinite computational resources).

We obtained these results by exploiting symmetrization procedures to model structural assessments of indistinguishability. This approach, which is called "first quantization" in QM, has a main drawback: it includes redundant information. More specifically, it potentially allows us to gamble on the state of a single particle which is not a physical observable (it is impossible in the first place to tell which particle is which). This constitutes a well-known limit in QM. As an example, we had to impose the condition on physical observables given by Equation (14) in order to obtain Proposition 22.

In QM, there is another formalism to work with indistinguishable particles, called *second quantization*. Its language allows one to ask the following question "How many particles are there in each state?". Since this formalism does not refer to the labelling of particles, it contains no redundant information. As future work, we plan to provide a gambling formulation of QM for the *second quantization*, exploring the connection with the count vectors formalism developed by De Cooman and Quaeghebeur (2012).

Author Contributions

All authors have contributed equally to the manuscript.

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