Supplementary Material: Proofs

Proof [Proof of Lemma 8] We will show that (i) $K_{\mathscr{M}} \subseteq \bigcap\{K_{D_p}\colon p \in \mathscr{M}\}$ and (ii) $K_{\mathscr{M}} \supseteq \bigcap\{K_{D_p}\colon p \in \mathscr{M}\}$. For (i), consider any A in $K_{\mathscr{M}}$, meaning that $A \cap \mathscr{L}_{>0} \neq \emptyset$ or $(\forall p \in \mathscr{M})(\exists f \in A)E_p(f) > 0$. Both cases imply that $A \cap D_p \neq \emptyset$ for every p in $\mathscr{M}$, whence indeed $A \in \bigcap\{K_{D_p}\colon p \in \mathscr{M}\}$. For (ii), consider any A in $\bigcap\{K_{D_p}\colon p \in \mathscr{M}\}$, meaning that $A \cap D_p \neq \emptyset$ for all p in $\mathscr{M}$, and hence indeed $A \in K_{\mathscr{M}}$. ∎

Proof [Proof of Proposition 19] We will show that (i) $K_{\bigotimes_{j=1}^n D_j} \subseteq \bigotimes_{j=1}^n K_{D_j}$ and (ii) $K_{\bigotimes_{j=1}^n D_j} \supseteq \bigotimes_{j=1}^n K_{D_j}$.

For (i), consider any A in $K_{\bigotimes_{j=1}^n D_j}$. Then $A \cap \bigotimes_{j=1}^n D_j \neq \emptyset$, so let $f \in A$ belong to $\bigotimes_{j=1}^n D_j$. Then $f \in \mathscr{L}(\mathscr{X}_{1:n})_{>0}$—in which case $A \in \bigotimes_{j=1}^n K_{D_j}$ by coherence—or $f \geq \sum_{k=1}^m \lambda_k f_k$ for some m in $\mathbb{N}$, $f_1, \dots, f_m$ in $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$ and m real coefficients $\lambda_{1:m} > 0$. But then, for every k in $\{1,\dots,m\}$, the gamble set $A_k := \{f_k\}$ belongs to $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$. Let furthermore $\lambda_{1:m}^{f_{1:m}} := \lambda_{1:m} > 0$ for the unique—and hence all—$f_{1:m}$ in $\bigtimes_{k=1}^m A_k$. This implies that $\{\sum_{k=1}^m f_k\} = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k\colon f_{1:m} \in \bigtimes_{k=1}^m A_k\}$ belongs to $\operatorname{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}})$ and since $f \geq \sum_{k=1}^m f_k$, also $\{f\} \in \operatorname{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0})$. Since $f \in A$, we have that then indeed $A \in \bigotimes_{j=1}^n K_{D_j}$.

For (ii), consider any A in $\bigotimes_{j=1}^n K_{D_j}$. Then $A \supseteq B \setminus \mathscr{L}(\mathscr{X}_{1:n})_{\leq 0}$ for some B in $\operatorname{Posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0})$, meaning that $B = \{\sum_{k=1}^m \lambda_k^{f_{1:m}} f_k\colon f_{1:m} \in \bigtimes_{k=1}^n B_k\}$ for some m in $\mathbb{N}$, $B_1, \dots, B_m$ in $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}$ and, for every $f_{1:m}$ in $\bigtimes_{k=1}^m B_k$, m real coefficients $\lambda_{1:m}^{f_{1:m}} > 0$. For any k in $\{1,\dots,m\}$ we have that B_k belongs to $\mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}$—in which case we call $B_k := \{g_k\}$—or $B_k = \mathbb{I}_E B'_k$ for some j in $\{1,\dots,n\}$, E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n} \setminus \{j\})$ and B'_k in K_{D_j}, meaning that $B'_k \cap D_j \neq \emptyset$—in which case we let h_k belong to $B'_k \cap D_j$ and define $g_k := \mathbb{I}_E h_k$. Then the gamble $f := \sum_{k=1}^m \lambda_k^{g_{1:m}} g_k$ belongs to B, and all of its terms $\lambda_k^{g_{1:m}} g_k$ either are equal to 0, or belong to $\mathscr{L}(\mathscr{X}_{1:m})_{>0}$ or to $\bigcup_{j=1}^n A_{1:n\setminus\{j\}\to\{j\}}$. Since not all of these terms are equal to 0, by Theorem 18 then $f \in \bigotimes_{j=1}^n D_j$, so that B belongs to $K_{\bigotimes_{j=1}^n D_j}$, and therefore indeed so does A. ∎

Proof [Proof of Theorem 20] This proof will consist of five parts: we will subsequently show that (i) $\bigotimes_{j=1}^n K_j$ is coherent, (ii) it is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, (iii) $\operatorname{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$ for every ℓ in $\{1,\dots,n\}$, (iv) $\bigotimes_{j=1}^n K_j$ is epistemically independent, and (v) $\bigotimes_{j=1}^n K_j$ is the smallest such set of desirable gamble sets. Then (i), (iii) and (iv) establish that $\bigotimes_{j=1}^n K_j$ is an independent product of $K_1, \dots, K_n$, which is by (v) the smallest one. (ii) establishes the last claim about $\bigotimes_{j=1}^n K_j$'s representation.

For (i), to show that $\bigotimes_{j=1}^n K_j$ is coherent, we will regard $\mathscr{A} := \bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$ as an assessment on $\mathscr{Q}(\mathscr{X}_{1:n})$. By Theorem 9 it suffices to show that $\mathscr{A} \subseteq K_D$ for some coherent set of desirable gambles $D \subseteq \mathscr{L}(\mathscr{X}_{1:n})$—in other words, that $\mathscr{A}$ is consistent.

To this end, note already using Theorem 7 that $\mathbf{D}(K_1), \dots, \mathbf{D}(K_n)$ all are non-empty since $K_1, \dots, K_n$ are coherent. Consider any D_1 in $\mathbf{D}(K_1), \dots, D_n$ in $\mathbf{D}(K_n)$, and let $D^* := \bigotimes_{j=1}^n D_j$. Then Theorem 18 implies that D^* is a coherent set of desirable gambles on $\mathscr{L}(\mathscr{X}_{1:n})$ that is epistemically independent—by which we mean that $\operatorname{marg}_O D^* = \operatorname{marg}_O(D^* \rfloor E_I)$ for all disjoint non-empty subsets I and O of $\{1,\dots,n\}$ and E_I in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_I)$—and marginalizes to $D_1, \dots, D_n$. We will show that $\mathscr{A} \subseteq K_{D^*}$. To this end, consider any A in $\mathscr{A}$, meaning that there is an index j in $\{1,\dots,n\}$ such that $A \in \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$, or, in other words, such that $A = \mathbb{I}_E B$ for some B in K_j and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. Since D_j belongs to $\mathbf{D}(K_j)$ we have that $K_j \subseteq K_{D_j}$, and therefore $B \in K_{D_j} = K_{\operatorname{marg}_j D^*}$. Since $K_{\operatorname{marg}_j D^*} = \operatorname{marg}_j K_{D^*}$ by Proposition 15, this means that $B \in K_{D^*}$. But D^* is an epistemically independent set of desirable gambles, and it therefore satisfies $\operatorname{marg}_j(D^* \rfloor E) = \operatorname{marg}_j D^*$, or in other words, $f \in D^* \Leftrightarrow \mathbb{I}_E f \in D^*$, for any f in $\mathscr{L}(\mathscr{X}_j)$, and hence also $A = \mathbb{I}_E B \in K_{D^*}$. Since the choice of A in $\mathscr{A}$ was arbitrary, this implies that indeed $\mathscr{A} \subseteq K_{D^*}$, guaranteeing that indeed $\bigotimes_{j=1}^n K_j$ is coherent.

For (ii), since we have just proved that $\mathscr{A}$ is consistent, we know by Theorem 9 that

$$
\begin{aligned}
\bigotimes_{j=1}^n K_j &= \bigcap\{K_D\colon D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } \mathscr{A} \subseteq K_D\} \\
&= \bigcap\{K_D\colon D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } (\forall j \in \{1,\dots,n\})\mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_D\} \\
&= \bigcap\{K_D\colon D \in \mathbf{D}(\mathscr{X}_{1:n}) \text{ and } (\forall j \in \{1,\dots,n\}, B \in K_j, E \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}}))\mathbb{I}_E B \in K_D\} = \bigcap\{K_D\colon D \in \mathbf{D}^*\},
\end{aligned}
$$

where we defined $\mathbf{D}^* := \{D \in \mathbf{D}(\mathscr{X}_{1:n})\colon (\forall j \in \{1,\dots,n\}, B \in K_j, E \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}}))\mathbb{I}_E B \in K_D\}$ for the sake of brevity. This collection $\mathbf{D}^*$ has two interesting properties: it satisfies $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_{D^*}$ for every D^* in $\mathbf{D}^*$, as can be seen from its definition. It also satisfies for every j in $\{1,\dots,n\}$ the inclusion $\operatorname{marg}_j \mathbf{D}^* \subseteq \mathbf{D}(K_j)$—in other words, $\operatorname{marg}_j D^* \in \mathbf{D}(K_j)$ for all D^* in $\mathbf{D}^*$. To show this last property, consider any D^* in $\mathbf{D}^*$, j in $\{1,\dots,n\}$, and consider $E := \mathscr{X}_{1:n\setminus\{j\}} \in \mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. That D^* belongs to $\mathbf{D}^*$ implies that $B = \mathbb{I}_E B \in K_{D^*}$ for every B in K_j, and hence $K_j \subseteq K_{D^*}$.

But K_j is a set of desirable gamble sets on $\mathscr{X}_j$, so $K_j \subseteq \mathrm{marg}_j K_{D^*} = K_{\mathrm{marg}_j D^*}$, where the equality is due to Proposition 15. This implies that indeed $\mathrm{marg}_j D^* \in \mathbf{D}(K_j)$.

This part of the proof is established if we show that $\bigcap\{K_D\colon D \in \mathbf{D}^*\} = \bigcap\{K_D\colon D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$. We will first show that $\bigotimes_{j=1}^n \mathbf{D}(K_j) \subseteq \mathbf{D}^*$. To this end, consider any D in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$—meaning that $D = \bigotimes_{j=1}^n D_j$ for some D_1 in $\mathbf{D}(K_1)$, $\dots$, D_n in $\mathbf{D}(K_n)$—and any j in $\{1,\dots,n\}$, B in K_j and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. That D_j belongs to $\mathbf{D}(K_j)$ implies that $B \in K_{D_j}$, and Theorem 18 tells us that $\mathrm{marg}_j D = D_j$, so $B \in K_D$. But then $f \in D$ for some f in B, and since D is an epistemically independent set of desirable gambles, therefore $\mathbb{I}_E f \in D$, whence $\mathbb{I}_E B \in K_D$. This implies that $D \in \mathbf{D}^*$, and therefore, since the choice of D in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ was arbitrary, indeed $\bigotimes_{j=1}^n \mathbf{D}(K_j) \subseteq \mathbf{D}^*$, which implies that $\bigcap\{K_D\colon D \in \mathbf{D}^*\} \subseteq \bigcap\{K_D\colon D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$.

To establish the equality between these two intersections, it suffices to prove that also the converse set inclusion holds. To this end, consider any A in $\bigcap\{K_D\colon D \in \bigotimes_{j=1}^n \mathbf{D}(K_j)\}$, meaning that $A \cap \bigotimes_{j=1}^n D_j \neq \emptyset$ for all D_1 in $\mathbf{D}(K_1), \dots, D_n$ in $\mathbf{D}(K_n)$. We need to show that then $A \in \bigcap\{K_D\colon D \in \mathbf{D}^*\}$—or in other words, that $A \cap D^* \neq \emptyset$ for any D^* in $\mathbf{D}^*$—so consider any D^* in $\mathbf{D}^*$. We have established earlier that then $\mathrm{marg}_j D^* \in \mathbf{D}(K_j)$ for any j in $\{1,\dots,n\}$, so that $\bigotimes_{j=1}^n \mathrm{marg}_j D^*$ belongs to $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ and we therefore have that $A \cap \bigotimes_{j=1}^n \mathrm{marg}_j D^* \neq \emptyset$, or in other words, that $A \in K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*}$. But we have seen in Proposition 19 that $K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*}$ is the smallest element of $\overline{\mathscr{K}}$ that includes $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}}$, and therefore, since we already have established above that $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K_{D^*}$, we have that $K_{\bigotimes_{j=1}^n \mathrm{marg}_j D^*} \subseteq K_{D^*}$. This implies that $A \in K_{D^*}$, whence indeed $A \cap D^* \neq \emptyset$.

For (iii), consider any ℓ in $\{1,\dots,n\}$, and we will show that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$. We know from the second part of this proof, established above, that $\bigotimes_{j=1}^n K_j$ is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, and therefore also, using Proposition 15, that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j)$ is represented by $\mathrm{marg}_\ell(\bigotimes_{j=1}^n \mathbf{D}(K_j))$. Infer the following chain of equalities:

$$\begin{aligned}
\mathrm{marg}_\ell\left(\bigotimes_{j=1}^n \mathbf{D}(K_j)\right) &= \mathrm{marg}_\ell\left(\left\{\bigotimes_{j=1}^n D_j\colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\right\}\right)\\
&= \left\{\mathrm{marg}_\ell\left(\bigotimes_{j=1}^n D_j\right)\colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\right\}\\
&= \{D_\ell\colon D_1 \in \mathbf{D}(K_1),\dots,D_n \in \mathbf{D}(K_n)\} = \mathbf{D}(K_\ell),
\end{aligned}$$

where the first equality follows from the definition of $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, the second one from the definition above Proposition 15 of $\mathrm{marg}_O(\mathbf{D})$ for any collection $\mathbf{D}$ of sets of desirable gambles, and the third one from Theorem 18. This means that $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j)$ is represented by $\mathbf{D}(K_\ell)$. Theorem 7 then implies that indeed $\mathrm{marg}_\ell(\bigotimes_{j=1}^n K_j) = K_\ell$.

Finally, for (iv), let $K^* \subseteq \mathscr{Q}(\mathscr{X}_{1:n})$ be the smallest independent product of K_1, $\dots$, K_n. Since K^* is epistemically independent, we have by Equation (4) in particular, for any j in $\{1,\dots,n\}$ and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$, that

$$\mathrm{marg}_j(K^* \rfloor E) = \mathrm{marg}_j K^* = K_j,$$

where the first equality holds because K is epistemically independent, and the second one because K^* marginalizes to K_1, $\dots$, K_n. This implies that any B in K_j should belong to $K^* \rfloor E$, and hence that $\mathbb{I}_E B \in K^*$. Since this should hold for any j in $\{1,\dots,n\}$, B in K_j, and E in $\mathscr{P}_{\overline{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$, we have that $\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \subseteq K^*$. Since K^* is coherent, also $\mathrm{posi}(\bigcup_{j=1}^n \mathscr{A}_{1:n\setminus\{j\}\to\{j\}} \cup \mathscr{L}^{\mathrm{s}}(\mathscr{X}_{1:n})_{>0}) \subseteq K^*$. But this tells us that $\bigotimes_{j=1}^n K_i \subseteq K^*$, establishing that $\bigotimes_{j=1}^n K_i$ indeed is the smallest independent product of K_1, $\dots$, K_n. ■

Proof [Proof of Proposition 21] By Theorem 20 $\bigotimes_{\ell\in L_1\cup L_2} K_\ell$ is represented by $\bigotimes_{\ell\in L_1\cup L_2} \mathbf{D}(K_\ell)$. Infer using the associativity of the independent natural extension for sets of desirable gambles that

$$\begin{aligned}
\bigotimes_{\ell\in L_1\cup L_2} \mathbf{D}(K_\ell) &= \left\{\bigotimes_{\ell\in L_1\cup L_2} D_\ell\colon (\forall \ell \in L_1\cup L_2) D_\ell \in \mathbf{D}(K_\ell)\right\}\\
&= \left\{\bigotimes_{\ell_1\in L_1} D_{\ell_1} \otimes \bigotimes_{\ell_2\in L_2} D_{\ell_2}\colon (\forall \ell \in L_1\cup L_2) D_\ell \in \mathbf{D}(K_\ell)\right\}\\
&= \left\{\bigotimes_{\ell_1\in L_1} D_{\ell_1}\colon (\forall \ell_1 \in L_1) D_{\ell_1} \in \mathbf{D}(K_{\ell_1})\right\} \otimes \left\{\bigotimes_{\ell_2\in L_2} D_{\ell_2}\colon (\forall \ell_2 \in L_1\cup L_2) D_{\ell_2} \in \mathbf{D}(K_{\ell_2})\right\}\\
&= \bigotimes_{\ell_1\in L_1} \mathbf{D}(K_{\ell_1}) \otimes \bigotimes_{\ell_2\in L_2} \mathbf{D}(K_{\ell_2}),
\end{aligned}$$

so that $\bigotimes_{\ell\in L_1\cup L_2} K_\ell$ is represented by the independent natural extension $\bigotimes_{\ell_1\in L_1}\mathbf{D}(K_{\ell_1})\otimes\bigotimes_{\ell_2\in L_2}\mathbf{D}(K_{\ell_2})$ of two coherent sets of desirable gambles $\bigotimes_{\ell_1\in L_1}\mathbf{D}(K_{\ell_1})$ and $\bigotimes_{\ell_2\in L_2}\mathbf{D}(K_{\ell_2})$. Theorem 20 then implies that indeed $\bigotimes_{\ell\in L_1\cup L_2} K_\ell = \bigotimes_{\ell_1\in L_1} K_{\ell_1}\otimes\bigotimes_{\ell_2\in L_2} K_{\ell_2}$. ■

Proof [Proof of Proposition 22] Use Theorem 20 to infer that $\bigotimes_{j=1}^n K_j$ is represented by $\bigotimes_{j=1}^n \mathbf{D}(K_j)$, so $\bigotimes_{j=1}^n K_j = \bigcap\{K_D\colon D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\}$. Note that, by Theorem 18, any D in $\bigotimes_{j=1}^n \mathbf{D}(K_j)$ satisfies

$$\mathrm{marg}_O D = \mathrm{marg}_O(D\rfloor E_I)$$

for any disjoint non-empty subsets I and O of $\{1,\dots,n\}$, and E_I in $\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_I)$. Consider any A in $\mathscr{Q}(\mathscr{X}_I)$ and $E_A\subseteq\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_O)$, and infer the following equivalences

$$A\in\bigotimes_{j=1}^n K_j \Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)(\exists f\in A)f\in D \Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)(\exists f\in A)\mathbb{I}_{E_f}f\in D$$
$$\Leftrightarrow \Big(\forall D\in\bigotimes_{j=1}^n\mathbf{D}(K_j)\Big)E_A A\cap D\neq\emptyset \Leftrightarrow E_A A\in\bigotimes_{j=1}^n K_j,$$

which establishes that $\bigotimes_{j=1}^n K_j$ satisfies the stronger requirement of Equation (9).

To show that then, as a consequence, any independent product of K_1, …, K_n includes $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}} \coloneqq \bigcup_{j=1}^n\{E_A\cdot A\colon A\in K_j \text{ and } E_A\subseteq\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})\}$, it suffices to show that the smallest independent product $\bigotimes_{j=1}^n K_j$ of K_1, …, K_n includes $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}}$. To this end, consider any j in $\{1,\dots,n\}$ and any A in K_j. Then since $\bigotimes_{j=1}^n K_j$ marginalizes to K_j, we have $A\in\bigotimes_{j=1}^n K_j$. By Equation (9) [use $O\coloneqq\{j\}$ and $I\coloneqq\{1,\dots,n\}\setminus\{j\}$], then also $E_A\cdot A\in\bigotimes_{j=1}^n K_j$ for any $E_A\subseteq\mathscr{P}_{\bar{\emptyset}}(\mathscr{X}_{1:n\setminus\{j\}})$. Since the choice of j in $\{1,\dots,n\}$ was arbitrary, this implies that indeed $\mathscr{A}^*_{1:n\setminus\{j\}\to\{j\}}\subseteq\bigotimes_{j=1}^n K_j$. ■

Proof [Proof of Lemma 24] To show that D is coherent, it suffices by Theorem 6 to show that $\mathscr{L}_{\leq 0}\cap\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2)=\emptyset$. To this end, consider any f in $\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2)$, meaning that $f=\sum_{k=1}^m\lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m}>0$, and gambles f_1, …, f_m in $\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2$. For every k in $\{1,\dots,m\}$, if f_k belongs to $\mathbb{I}_{\{F\}}D_1$ then $f_k(U,H)=f_k(U,T)=0$ and $f_k(F,H)+f_k(F,T)>0$, and if f_k belongs to $\mathbb{I}_{\{U\}}D_2$ then $f_k(F,H)=f_k(F,T)=0$ and $f_k(U,H)+f_k(U,T)>0$, or $f_k(U,H)+f_k(U,T)=0$ but then $f_k(U,H)>f_k(U,T)$. This implies that $f(\bullet,H)+f(\bullet,T)>0$ whence indeed $f\notin\mathscr{L}_{\leq 0}$.

To show that it is no independent product, let us show that $\mathrm{marg}_Y D\subset\mathrm{marg}_Y(D\rfloor\{U\})$, so that learning that the coin is unfair, results in a bigger Y-marginal than not learning anything at all. More specifically, we will show that $\mathrm{marg}_Y D=D_1$ and $\mathrm{marg}_Y(D\rfloor\{U\})=D_2$.

To show that $\mathrm{marg}_Y D\subseteq D_1$, consider any f in $\mathrm{marg}_Y D$. Then $f\in\mathscr{L}(\mathscr{Y})$ and $f\in D$, meaning that $f>0$—in which case $f\in D_1$ by its coherence—or $f\geq\sum_{k=1}^m\lambda_k f_k$ for some m in $\mathbb{N}$, real coefficients $\lambda_{1:m}>0$, and gambles f_1, …, f_m in $\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_2$. Since f belongs to $\mathscr{L}(\mathscr{Y})$, we have that $f\geq\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,\bullet)$, and therefore $f(H)+f(T)\geq\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,H)+\frac{1}{2}\sum_{x\in\mathscr{X}}\sum_{k=1}^m\lambda_k f_k(x,T)>0$, so that indeed $f\in D_1$.

That also $\mathrm{marg}_Y D\supseteq D_1$ follows once we realise that $D_1\subseteq D_2$, whence $D\supseteq\mathrm{posi}(\mathbb{I}_{\{F\}}D_1\cup\mathbb{I}_{\{U\}}D_1\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=\mathrm{posi}(D_1\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})$, which is the *cylindrical extension*[12] of D_1, a coherent set of desirable gambles that marginalizes to D_1.

To show now that conditioning on $\{U\}$ changes the marginal $\mathrm{marg}_Y(D\rfloor\{U\})$ information to D_2, let us show first that $\mathrm{marg}_Y(D\rfloor\{U\})\subseteq D_2$. This follows once we realise that $D_1\subseteq D_2$ and therefore $D\subseteq\mathrm{posi}(\mathbb{I}_{\{F\}}D_2\cup\mathbb{I}_{\{U\}}D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=\mathrm{posi}(D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})$, which is the cylindrical extension of D_2, a coherent set of desirable gambles that marginalizes to D_2. This implies that $\mathrm{marg}_Y D\subseteq\mathrm{marg}_Y\,\mathrm{posi}(D_2\cup\mathscr{L}(\mathscr{X}\times\mathscr{Y})_{>0})=D_2$.

To show, conversely, that $\mathrm{marg}_Y(D\rfloor\{U\})\supseteq D_2$, consider any f in D_2. This implies that $\mathbb{I}_{\{U\}}f\in\mathbb{I}_{\{U\}}D_2\subseteq D$. By the conditioning rule for sets of desirable gambles

$$D\rfloor E\coloneqq\{f\in\mathscr{L}(E)\colon\mathbb{I}_E f\in D\},$$

then $f\in D\rfloor\{U\}$, and since f belongs to $\mathscr{L}(\mathscr{Y})$, indeed $f\in\mathrm{marg}_Y(D\rfloor\{U\})$. ■

Proof [Proof of Lemma 25] Since Y is a binary variable, it suffices to check that

$$\{\mathbb{I}_{\{U\}}f+\varepsilon,-\mathbb{I}_{\{F\}}f+\varepsilon\}\in K_D$$

12. See De Cooman and Miranda [11, Proposition 7].

for all f in $\mathscr{L}(\mathscr{X})$ and $\varepsilon \in \mathbb{R}_{>0}$, as discussed right after Definition 23. So consider any f in $\mathscr{L}(\mathscr{X})$ and $\varepsilon \in \mathbb{R}_{>0}$; we need to show that then $\mathbb{I}_{\{F\}}f+\varepsilon$ or $-\mathbb{I}_{\{U\}}f+\varepsilon$ belongs to D. We will proceed by considering two exhaustive cases: (i) $f \in D$ and (ii) $f \notin D$.

For (i) $f \in D$ implies that $\mathbb{I}_{\{U\}}f = \mathbb{I}_{\{U\}}f(U,\bullet) \in D\rfloor\{U\}$. But in the proof of Lemma 24 we have established that $\mathrm{marg}_Y D\rfloor\{U\} = D_2$, and therefore $f(U,\bullet) \in D_2$, whence $\mathbb{I}_{\{U\}}f = \mathbb{I}_{\{U\}}f(U,\bullet) \in \mathbb{I}_{\{U\}}D_2 \subseteq D$, and therefore indeed $\mathbb{I}_{\{U\}}f+\varepsilon \in D$.

For (ii) $f \notin D$ implies that $\mathbb{I}_{\{F\}}f = \mathbb{I}_{\{F\}}f(F,\bullet) \notin D\rfloor\{F\}$. By a completely similar argument as in the proof of Lemma 24, we can establish that $\mathrm{marg}_Y D\rfloor\{F\} = D_1$, so that $\mathbb{I}_{\{F\}}f(F,\bullet) \notin D_1$. But this means that $\mathbb{I}_{\{F\}}f(F,H)+\mathbb{I}_{\{F\}}f(F,T) \leq 0$, whence $-\mathbb{I}_{\{F\}}f(F,H)+\varepsilon-\mathbb{I}_{\{F\}}f(F,T)+\varepsilon > 0$ and therefore indeed $-\mathbb{I}_{\{F\}}f+\varepsilon = -\mathbb{I}_{\{F\}}f(F,\bullet)+\varepsilon \in D_1 \subseteq D$. ■