CREPO: An Open Repository to Benchmark Credal Network Algorithms (Supplementary Material)

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Appendix A. Supplementary Material

Consider a closed and convex set C in $\mathbb{R}^d$. Let d denote also the dimension of C. The vertices of C are assumed to be finite and denoted as $e(C)$. A *hyperplane* H in $\mathbb{R}^d$ can be parametrized by a pair $(\boldsymbol{v}, w)$, with $\boldsymbol{v} \in \mathbb{R}^d$ and $w \in \mathbb{R}$ as follows:

$$H_{\boldsymbol{v},w} := \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{v} \cdot \boldsymbol{x} = w\}\,. \tag{1}$$

The *segment* S connecting points $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ is instead:

$$S_{\boldsymbol{a},\boldsymbol{b}} := \{\boldsymbol{x} \in \mathbb{R}^d := \lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b}, 0 \leq \lambda \leq 1\}\,. \tag{2}$$

Definition 1 *$H_{\boldsymbol{v},w}$ is a* supporting *hyperplane for C passing through $\boldsymbol{x}^* \in C$ if and only if $\boldsymbol{x}^* \in P_{\boldsymbol{v},w}$ and $\boldsymbol{v} \cdot \boldsymbol{x} \leq \boldsymbol{v} \cdot \boldsymbol{x}^*$ for each $\boldsymbol{x} \in C$.*

Definition 2 *A point $\boldsymbol{x}^*$ belongs to the* boundary *$b(C)$ of the convex set C if and only if there is at least a supporting hyperplane for C passing through $\boldsymbol{x}^*$.*

We use notation CH for the convex hull of a set of points, e.g., $C := \mathrm{CH}[e(C)]$. The following result holds.

Lemma 3 *Let C be a convex set in $\mathbb{R}^d$ such that d is also its dimension (i.e., Given $\boldsymbol{a}, \boldsymbol{b} \in e(C)$, let $\boldsymbol{x}^* := \frac{1}{2}(\boldsymbol{a} + \boldsymbol{b})$. Let also $C' := \mathrm{CH}[e(C) \setminus \{\boldsymbol{a}, \boldsymbol{b}\}]$, while $H_{\boldsymbol{v},w}$ denotes a supporting hyperplane for C through $\boldsymbol{x}^*$. It holds that, if $S_{\boldsymbol{a},\boldsymbol{b}} \subset H_{\boldsymbol{v},w}$, then $\boldsymbol{x}^* \notin C'$.*

Proof *By construction $\boldsymbol{x} \in S_{\boldsymbol{a},\boldsymbol{b}}$. Assume,* ad absurdum, *$\boldsymbol{x}^* \in C'$. Thus, $\boldsymbol{x}^*$ should be a convex combination of the vertices of C', i.e.*

$$\boldsymbol{x}^* = \sum_{z \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_z \boldsymbol{z}\,, \tag{3}$$

where $\lambda_{\boldsymbol{z}} \geq 0$ for each $\boldsymbol{z} \in e(C) \setminus \{\boldsymbol{a}, \boldsymbol{b}\}$ and $\sum_{\boldsymbol{z} \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_{\boldsymbol{z}} = 1$. Take the scalar product by $\boldsymbol{v}$:

$$\boldsymbol{v} \cdot \boldsymbol{x}^* = \boldsymbol{v} \cdot \left[\sum_{z \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_z \boldsymbol{z} \right] = \sum_{z \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_z \boldsymbol{v} \cdot \boldsymbol{z}\,. \tag{4}$$

By supporting hyperplane definition and simple algebra:

$$\boldsymbol{v} \cdot \boldsymbol{x}^* = \sum_{z \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_z \boldsymbol{v} \cdot \boldsymbol{z} \leq \sum_{z \in e(C) \setminus \{\boldsymbol{a},\boldsymbol{b}\}} \lambda_z \boldsymbol{v} \cdot \boldsymbol{x}^* = \boldsymbol{v} \cdot \boldsymbol{x}^*\,. \tag{5}$$

This implies $\boldsymbol{z} \in H_{\boldsymbol{v},w}$ for each $z \in e(C) \setminus \{\boldsymbol{a}, \boldsymbol{b}\}$. As also $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $H_{\boldsymbol{v},w}$, we have $e(C) \subset H_{\boldsymbol{v},w}$. In other words C is included in a hyperplane and it coincides with its boundary, but this is against the original assumption about the dimension of C. ■

As a consequence of this lemma we have that, in Definition 1, the midpoint of the two vertices added to C is a vertex of the new set. This simply following that the two points at minimum (Euclidean) distance belong to a same edge of a convex polytope and the credal set can be always parametrized in order to have full dimension (see discussion in Section 3. When coping with non-Euclidean distances, to have the same result, the two points at minimum distance in Definition 1 should be detected with the additional condition of belonging to a same edge.