Appendix A. Proofs of the main results

Proposition 21 *Consider a set of gambles $\mathscr{D} \subseteq \mathscr{L}$.*

If it is closed under the supremum norm topology, then it satisfies D4. Vice versa, if $\mathscr{D}$ satisfies also the following property:

$$f \geq g,\ g \in \mathscr{D} \Rightarrow f \in \mathscr{D} \tag{22}$$

then D4 implies closure in the supremum norm topology.

Proof It is well-known that $\mathscr{L}$ is a Banach space under the supremum norm and it is a linear topological space (with finite dimension n in our case) under the topology generated by the supremum norm (see [30]).

Now, consider $\mathscr{D}$ closed under the supremum norm topology. Then, the limit of every convergent sequence $(f_n)_{\{n\in\mathbb{N}\}}$ (respect to the supremum norm) with $f_n \in \mathscr{D}$ for every n, must be contained in $\mathscr{D}$. Consider then, a gamble f such that $f+\delta \in \mathscr{D}$ for every $\delta > 0$, then $f+\frac{1}{n} \in \mathscr{D}$ for every $n \in \mathbb{N}^*$. Its limit w.r.t. the supremum norm is f and, from the closure of $\mathscr{D}$, we know that $f \in \mathscr{D}$.

On the other hand, suppose $\mathscr{D}$ satisfies D4 and (22). Let us consider a succession $(f_n)_{\{n\in\mathbb{N}\}} \in \mathscr{D}$ convergent w.r.t. the sumpremum norm to a gamble $f \in \mathscr{L}$. We know that for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\sup|f_n - f| < \varepsilon$ for all $n \geq N$. In particular, this means that there exist $h \in \mathscr{L}$ such that:

$$f_n - f = h^+ - h^-,\ \sup|h| < \varepsilon \tag{23}$$

hence:

$$f = (f_n + h^-) - h^+ \tag{24}$$

but, $f_n + h^- \in \mathscr{D}$ by hypothesis, and $f = (f_n+h^-) - h^+ \geq (f_n+h^-) - \varepsilon$. Then $f+\varepsilon \geq (f_n+h^-) \in \mathscr{D}$, from which it follows that $f+\varepsilon \in \mathscr{D}$. This procedure can be repeated for every $\varepsilon > 0$. Then by D4, we have $f \in \mathscr{D}$. ∎

Proof [Proof of Proposition 3] Consider a pair of finite sets $(\mathscr{A},\mathscr{R})$ for which there exists a coherent set of gambles $\mathscr{D}$, such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Then, the minimal coherent set $\mathscr{D}$ that satisfies these conditions is $\overline{\mathscr{E}(\mathscr{A})} := \overline{\text{posi}(\mathscr{A} \cup T)}$, where $\text{posi}(\mathscr{K}) := \left\{\sum_{j=1}^{r} \lambda_j f_j : f_j \in \mathscr{K}, \lambda_j > 0, r \geq 1\right\}$ for every $\mathscr{K} \subseteq \mathscr{L}(\Omega)$ and where $\overline{\mathscr{K}'}$ of a set $\mathscr{K}' \subseteq \mathscr{L}$ represents the closure of $\mathscr{K}'$ with respect to the supremum norm topology. In fact, $\mathscr{E}(\mathscr{A})$ is clearly the minimal set $\mathscr{D}$ that satisfies D1 - D3 such that $\mathscr{D} \supseteq \mathscr{A}$. Then, thanks to Proposition 21, $\overline{\mathscr{E}(\mathscr{A})}$ is the minimal coherent set $\mathscr{D}'$ such that $\mathscr{D}' \supseteq \mathscr{A}$ and clearly, by hypothesis, we know also that $\overline{\mathscr{E}(\mathscr{A})} \cap \mathscr{R} = \emptyset$. This fact is also well-known in literature [30].

However, $\overline{\mathscr{E}(\mathscr{A})}$, by definition, is a polyhedral (convex) cone [1, Definition 2.3.2]. Indeed $\overline{\mathscr{E}(\mathscr{A})}$ can be rewritten as:

$$\overline{\mathscr{E}(\mathscr{A})} = \overline{\text{posi}(\mathscr{A} \cup T)} =$$

$$C := \left\{g : g = \sum_{j=1}^{r} \lambda_j f_j,\ f_j \in (\mathscr{A} \cup \{\mathbb{I}_{\omega_i}\}_{i=1}^{n}), r \geq 1, \lambda_j \geq 0\right\}$$

where the last equality derives from the facts that: $\mathscr{E}(\mathscr{A}) = \text{posi}(\mathscr{A} \cup T)$ is generated by the finite set $(\mathscr{A} \cup \{\mathbb{I}_{\omega_i}\}_{i=1}^{n})$; C is already closed under the usual topology of $\mathbb{R}^n$ that coincides with the closure with respect to the supremum norm topology, for every topological space with n dimension [30, Appendix D]. The latter is true because, thanks to the Minkowsky-Weyl theorem [1], we know that C is an intersection of a finite number of closed halfspaces whose bounding hyperspaces pass through the origin:

$$C = \{g : g^T \beta_j \geq 0,\ j = 1,\dots,N\} \tag{25}$$

with $\beta_j \in \mathbb{R}^n$. This concludes this part of the proof since it tells us that there exists a binary piecewise linear classifier $PLC(\cdot)$ with parameters β_j, which classifies $\mathscr{A} \cup T \subseteq \overline{\mathscr{E}(\mathscr{A})} = C =: \{g \in \mathscr{L} : PLC(g) = 1\}$ as 1 and $(\mathscr{R} \cup F)$, that has empty intersection with C, as -1.

Vice versa, consider a piecewise linearly separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and a classifier $PLC(\cdot) \in \text{PLC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$. Then:

$$\{g : PLC(g) = 1\} = \{g : g^T \beta_j \geq 0,\ \text{for all } j = 1,..,N\} \tag{26}$$

for some $\beta_j \in \mathbb{R}^n$ such that $\beta_{ji} \geq 0$, $\sum_i \beta_{ji} = 1$, for all i, j (constraints on β_j easily follow from the fact that $PLC(\cdot)$ classifies T as 1). Hence there exists a linear prevision P_j, such that $P_j(g) = g^T \beta_j$, for all g, for all $j = 1,\dots,N$ [30, Section 2.8,Section 3.2]. Therefore we have:

$$\{g : PLC(g) = 1\} =$$
$$\{g : P_j(g) \geq 0, \text{for all} j = 1,..,N\} = \{g : \underline{P}(g) \geq 0\},$$

where $\underline{P} := \min_j \{P_j\}$ is a coherent lower prevision [30, Theorem 3.3.3]. Hence, $\mathscr{D} := \{g : PLC(g) = 1\}$ is a coherent set of gambles [30, Theorem 3.8.1].

In particular, we have also that $\mathscr{A} \subseteq \{g : PLC(g) = 1\} = \mathscr{D}$ and $\mathscr{R} \cap (\{g : PLC(g) = 1\} = \mathscr{D}) = \emptyset$ by hypotheses. ∎

Proof [Proof of Proposition 5] Consider a piecewise linearly separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and a classifier $PLC(\cdot) \in \text{PLC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\{\beta_j\}_{j=1}^{N}$.

Then, a classifier $LC_\phi(\cdot)$ of the type (5) with parameters $\omega_j = \beta_j$ and $\beta'_j = \beta_j$ for all $j = 1,\dots,N$, classifies $\mathscr{A} \cup T$ as 1 and $\mathscr{R} \cup F$ as -1. Indeed, consider $g \in \mathscr{L}$ and let us define $m := \min(g^T\beta_1,\dots,g^T\beta_N)$. Then:

$$\sum_{j=1}^{N} (\phi_j(g))^T \beta_j = \sum_{j=1}^{N} (\mathbb{I}_{\mathscr{B}_j}(g) g)^T \beta_j = \sum_{k=1}^{K} g^T \beta_k = Km,$$

where, for every j, $\mathscr{B}_j$ are the partitions of the type 4 with $\omega_j = \beta_j$ and $g^T\beta_k = m$, for all $k = 1,\dots,K$, with $1 \leq K \leq N$.

Hence, g is classified in the same way by the classifiers $PLC(\cdot)$ and $LC_\phi(\cdot)$. Therefore, in particular, if $g \in (\mathscr{A} \cup T)$, $m \geq 0$ and hence $\sum_{j=1}^{N}(\phi_j(g))^T\beta_j = Km \geq 0$, if instead $g \in (\mathscr{R} \cup F)$ then $m < 0$ and hence $\sum_{j=1}^{N}(\phi_j(g))^T\beta_j = Km < 0$.

Vice versa, let us consider a Φ-separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and let us suppose the existence of a classifier $LC_\phi(\cdot) \in \mathrm{LC}_\Phi(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\omega_j = \beta'_j$, for all $j = 1, ..., N$. Let us define $m' := \min(g^T\beta'_1, ..., g^T\beta'_N)$. Then, for any $g \in \mathscr{L}$ we have:

$$\sum_{j=1}^{N}(\phi_j(g))^T\beta'_j = \sum_{k=1}^{K} g^T\beta'_k = Km', \tag{27}$$

where again $g^T\beta'_k = m'$, for all $k = 1, \dots K$, with $1 \leq K \leq N$. Let us consider a binary piecewise linear classifier $PLC(\cdot)$ with parameters $\{\beta'_j\}_{j=1}^{N}$. Then, again, g is classified in the same way by the classifiers $LC_\phi(\cdot)$ and $PLC(\cdot)$. This is in particular true for $g \in \mathscr{A} \cup T$ and $g \in \mathscr{R} \cup F$. This means also that $\beta'_j \gneq 0$, for all $j = 1, .., N$.

■

Lemma 22 *If a set $\mathscr{D} \subseteq \mathscr{L}$, satisfies D1, D3* and D4 then it satisfies* (22).

Proof Consider $f \geq g$ with $g \in \mathscr{D}$. Then $f = g + t$ with $t \in T$. For any $\varepsilon > 0$, $f + \varepsilon = g + t + \varepsilon$. Moreover, we can always find $\lambda \in (0, 1)$ such that $\lambda g \leq g + \varepsilon$.

Therefore, we have $f + \varepsilon = \lambda g + (1 - \lambda)\frac{(g+\varepsilon-\lambda g)+t}{1-\lambda}$. Now, $g \in \mathscr{D}$ by hypothesis and $\frac{(g+\varepsilon-\lambda g)+t}{1-\lambda} \in T$, so $f + \varepsilon \in \mathscr{D}$. This can be repeated for every $\varepsilon > 0$, then $f + \varepsilon \in \mathscr{D}$ for all $\varepsilon > 0$ that implies, by D4, that $f \in \mathscr{D}$. ■

Lemma 23 *Given a pair of finite sets $(\mathscr{A}, \mathscr{R})$ for which there exists a convex coherent set of gambles $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$, then the minimal such set is $\mathscr{D} = \overline{\mathrm{ch}(\mathscr{A} \cup T)}$.*

Proof $\overline{\mathrm{ch}(\mathscr{A} \cup T)}$ satisfies D1 by definition and D3* [24, Theorem 6.2] and D4, thanks to Proposition 21.

Let us indicate with $\mathrm{D}(\mathscr{A}, \mathscr{R})$, the class of convex coherent sets of gambles $\mathscr{D}$ such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Thanks to Lemma 22 and Proposition 21, every $\mathscr{D} \in \mathrm{D}(\mathscr{A}, \mathscr{R})$, is a convex closed set (respect to the topology of $\mathbb{R}^n$ or equivalently respect to the supremum norm topology) that contains $(\mathscr{A} \cup T)$.

Given the fact that $\overline{\mathrm{ch}(\mathscr{A} \cup T)} \supseteq \mathscr{A} \cup T$ and, by definition, it is the intersection of all the closed (respect to the topology of $\mathbb{R}^n$ or equivalently respect to the supremum norm topology) and convex sets containing $(\mathscr{A} \cup T)$, we have that $\overline{\mathrm{ch}(\mathscr{A} \cup T)} \subseteq \mathscr{D}$, for all $\mathscr{D} \in \mathrm{D}(\mathscr{A}, \mathscr{R})$.

But, every $\mathscr{D} \in \mathrm{D}(\mathscr{A}, \mathscr{R})$, satisfies $\mathscr{D} \cap (\mathscr{R} \cup F) = \emptyset$. Therefore, $\overline{\mathrm{ch}(\mathscr{A} \cup T)} \cap (\mathscr{R} \cup F) = \emptyset$, and hence it is also the smallest set $\mathscr{D} \in \mathrm{D}(\mathscr{A}, \mathscr{R})$. This concludes the proof.
■

Lemma 24 *Consider a finite set $\mathscr{A} \subseteq \mathscr{L}$. Then:*

$\overline{\mathrm{ch}(\mathscr{A} \cup T)} = \mathrm{ch}^+(\mathscr{A} \cup \{0\}) := \{g : g \geq f,\ f \in \mathrm{ch}(\mathscr{A} \cup \{0\}\}$.

Proof First of all, we can observe that:

$$\begin{aligned}\mathrm{ch}^+(\mathscr{A} \cup \{0\}) = \{g : g \geq f,\ f \in \mathrm{ch}(\mathscr{A} \cup \{0\}\} =\\ = \sum_{i \in I} \alpha_i g_i + \sum_{j \in J} \gamma_j e_j =: \mathrm{ch}(\mathscr{A} \cup \{0\}) + \mathrm{posi}(e_1, \dots, e_n)\end{aligned}$$

with I, J finite, $g_i \in \mathscr{A} \cup \{0\}$, $\alpha_i, \gamma_i \geq 0$ and $\sum_i \alpha_i = 1$, where e_i is the canonical basis in $\mathbb{R}^n$ and $\mathrm{posi}(e_1, \dots, e_n)$ is a convex polyhedral cone. From [27, Corollary 7.1.b], it follows that $\mathrm{ch}^+(\mathscr{A} \cup \{0\})$ is a convex (closed) polyhedron. Hence $\overline{\mathrm{ch}^+(\mathscr{A} \cup \{0\})} = \mathrm{ch}^+(\mathscr{A} \cup \{0\})$. Now, we divide the proof in two parts.

- $\overline{\mathrm{ch}(\mathscr{A} \cup T)} \subseteq \mathrm{ch}^+(\mathscr{A} \cup \{0\})$. Notice that, thanks to the previous observation, it is sufficient to show that $\mathrm{ch}(\mathscr{A} \cup T) \subseteq \mathrm{ch}^+(\mathscr{A} \cup \{0\})$. So, let us consider $g \in \mathrm{ch}(\mathscr{A} \cup T)$. By definition, we have:

$$g = \sum_{k=1}^{r} \lambda_k g_k$$

  with $\lambda_k \geq 0$, for all $k = 1, ..., r$, $r \geq 1$, $\sum_{k=1}^{r} \lambda_k = 1$, $g_k \in (\mathscr{A} \cup T)$. Let us indicate with $Ind_{A \setminus T} := \{k \in \{1, ..., r\}$ such that : $g_k \in \mathscr{A} \setminus T\}$ and $Ind_T := \{k \in \{1, ..., r\}$ such that : $g_k \in T\}$. Then we have:

$$g \geq \sum_{k \in Ind_{A \setminus T}} \lambda_k g_k + \sum_{k \in Ind_T} \lambda_k 0,$$

  hence $g \in \mathrm{ch}^+(\mathscr{A} \cup \{0\})$.
- $\mathrm{ch}^+(\mathscr{A} \cup \{0\}) \subseteq \overline{\mathrm{ch}(\mathscr{A} \cup T)}$. By definition, $\overline{\mathrm{ch}(\mathscr{A} \cup T)}$ is a closed convex set that contains T. Therefore, from Proposition 21 and Lemma 22, we have:

$$\begin{aligned}\mathrm{ch}(\mathscr{A} \cup \{0\}) \subseteq \overline{\mathrm{ch}(\mathscr{A} \cup T)} \Rightarrow\\ \mathrm{ch}^+(\mathscr{A} \cup \{0\}) \subseteq \overline{\mathrm{ch}(\mathscr{A} \cup T)}.\end{aligned}$$

■

Proof [Proof of Proposition 8] Consider a pair of sets $(\mathscr{A}, \mathscr{R})$ for which there exists a convex coherent set of gambles $\mathscr{D}$, such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Then the minimal convex coherent set $\mathscr{D}$, which satisfies these conditions, is $\overline{\mathrm{ch}(\mathscr{A} \cup T)}$. Thanks to Lemma 24, we know that it can be rewritten as:

$$\overline{\mathrm{ch}(\mathscr{A} \cup T)} = \mathrm{ch}^+(\mathscr{A} \cup \{0\}), \tag{28}$$

where $\mathrm{ch}^+(\mathscr{A} \cup \{0\})$ is a convex polyhedron. Any convex polyhedron can be written as an intersection of hyperspaces, whose border is a piecewise affine function. Therefore, there exists a piecewise affine classifier $PAC(\cdot)$, such that $\overline{\text{ch}(\mathscr{A} \cup T)} = \text{ch}^+(\mathscr{A} \cup \{0\}) = \{g : PAC(g) = 1\}$. Note moreover that $\overline{\text{ch}(\mathscr{A} \cup T)} = \{g : PAC(g) = 1\} \supseteq (\mathscr{A} \cup T)$ and $(\overline{\text{ch}(\mathscr{A} \cup T)} = \{g : PAC(g) = 1\}) \cap (\mathscr{R} \cup F) = \emptyset$ by construction.

Vice versa, consider a piecewise affine separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$. Let us consider a piecewise affine classifier $PAC(\cdot) \in \text{PAC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$. Now, the set:

$$\mathscr{D} := \{g : PAC(g) = 1\} = \{g : g^T \beta_j + \alpha_j \geq 0, \text{ for all } j = 1, ..., N\}$$

for some $\beta_j \in \mathbb{R}^n$ with $\beta_j \gneq 0$ and $\alpha_j \in \mathbb{R}$ for all $j \in \{1, \ldots N\}$, is a convex coherent set of gambles such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Indeed:

- $T \subseteq \mathscr{D}$ and $\mathscr{D} \cap F = \emptyset$, by definition, hence it satisfies D1 and D2;
- $\mathscr{D}$ satisfies D3*. Consider $g_1, g_2 \in \mathscr{D}$. Then $tg_1 + (1-t)g_2 \in \mathscr{D}$, for all $t \in [0,1]$. Indeed,

$$(tg_1 + (1-t)g_2)^T \beta_j + \alpha_j = (tg_1)^T \beta_j + ((1-t)g_2)^T \beta_j + t\alpha_j + (1-t)\alpha_j = t((g_1)^T \beta_j + \alpha_j) + (1-t)(g_2^T \beta_j + \alpha_j) \geq 0$$

  for all $j \in \{1, ..., N\}$.
- $\mathscr{D}$ is closed in the usual topology of $\mathbb{R}^n$ because it is the intersection of a finite number of closed half-spaces hence, thanks to Proposition 21, it satisfies D4.

Clearly, by the fact that $PAC(\cdot) \in \text{PAC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$, it is also true that $\mathscr{A} \subseteq \mathscr{D}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. ■

Proof [Proof of Proposition 10]

Consider a piecewise affine separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and a classifier $PAC(\cdot) \in \text{PAC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\{\beta_j, \alpha_j\}_{j=1}^N$.

Then, a classifier $LC_\Psi(\cdot)$ of the type (11) with parameters $\omega'_j = \beta'_j = \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix}$, for all $j = 1, ..., N$, classifies $\mathscr{A} \cup T$ as 1 and $\mathscr{R} \cup F$ as -1. Indeed, consider $g \in \mathscr{L}$ and let us define $m := \min(g^T \beta_1 + \alpha_1, \ldots, g^T \beta_N + \alpha_N)$. Then:

$$\sum_{j=1}^N (\psi_j(g))^T \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix} = \sum_{j=1}^N \left(\mathbb{I}_{\mathscr{B}'_j} \left(\begin{bmatrix} g \\ 1 \end{bmatrix} \right) \begin{bmatrix} g \\ 1 \end{bmatrix} \right)^T \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix} = \sum_{k=1}^K (g^T \beta_k + \alpha_k) = Km,$$

where, for every j, $\mathscr{B}'_j$ are the partitions of the type 10 with $\omega'_j = \beta_j$ and $g^T \beta_k + \alpha_k = m$, for any $k = 1, ..., K$, with $1 \leq K \leq N$. Hence, g is classified in the same way by the classifiers $PAC(\cdot)$ and $LC_\Psi(\cdot)$. Therefore, in particular, if $g \in (\mathscr{A} \cup T)$, $m \geq 0$ and hence $\sum_{j=1}^N (\psi_j(g))^T \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix} = Km \geq 0$, if instead $g \in (\mathscr{R} \cup F)$ then $m < 0$ and hence $\sum_{j=1}^N (\psi_j(g))^T \begin{bmatrix} \beta_j \\ \alpha_j \end{bmatrix} < 0$.

Vice versa, let us consider a Ψ-separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and let us suppose the existence of a classifier $LC_\Psi(\cdot) \in \text{LC}_\Psi(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\omega'_j = \beta'_j$, for all $j = 1, ..., N$. Let us define $m' := \min(g^T \beta'_{1,1:n} + \beta'_{1,n+1}, ..., g^T \beta'_{N,1:n} + \beta'_{N,n+1})$. Then, for any $g \in \mathscr{L}$, we have:

$$\sum_{j=1}^N (\psi_j(g))^T \beta'_j = \sum_{k=1}^K (g^T \beta'_{k,1:n} + \beta'_{k,n+1}) = Km',$$

where $\beta'_{k,1:n}$ is the vector containing the first n components of β'_k, for every k, and where again $(g^T \beta'_k + \beta'_{k,n+1}) = m'$, for all $k = 1, \ldots K$, with $1 \leq K \leq N$. Let us consider a binary piecewise affine classifier $PAC(\cdot)$ with parameters $\{\beta'_{j,1:n}, \beta'_{j,n+1}\}_{j=1}^N$. Then, again, g is classified in the same way by the classifiers $LC_\Psi(\cdot)$ and $PAC(\cdot)$. This is in particular true for $g \in \mathscr{A} \cup T$ and $g \in \mathscr{R} \cup F$. This means also that $\beta'_{j,1:n} \gneq 0$, for all $j = 1, .., N$ and $\beta'_{j,n+1} \geq 0$, for all $j = 1, .., N$, with at least a $\beta'_{k,n+1} = 0$. ■

Lemma 25 *Given a pair of finite sets $(\mathscr{A}, \mathscr{R})$ for which there exists a positive additive coherent set of gambles $\mathscr{D}$, such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$, then the smallest such set is:*

$$\mathscr{D} = \uparrow (\mathscr{A} \cup \{0\}) := \{g : (\exists f \in \mathscr{A} \cup \{0\})\, g \geq f\}.$$

Proof $\uparrow (\mathscr{A} \cup \{0\})$ satisfies D1, D3** and $\mathscr{A} \subseteq \uparrow (\mathscr{A} \cup \{0\})$ by construction. Moreover, it satisfies also D4 by Proposition 21, because it is closed respect to the usual topology of $\mathbb{R}^n$ (it is a finite union of closed sets).

Let us indicate with $\text{P}(\mathscr{A}, \mathscr{R})$, the class of positive additive coherent sets of gambles $\mathscr{D}$, such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Clearly, each $\mathscr{D} \in \text{P}(\mathscr{A}, \mathscr{R})$ satisfies $\mathscr{D} \supseteq \uparrow (\mathscr{A} \cup \{0\})$. But, every $\mathscr{D} \in \text{P}(\mathscr{A}, \mathscr{R})$, satisfies also $\mathscr{D} \cap (\mathscr{R} \cup F) = \emptyset$. Therefore, $\uparrow (\mathscr{A} \cup \{0\}) \cap (\mathscr{R} \cup F) = \emptyset$. So, it is also the smallest positive additive coherent set of gambles $\mathscr{D} \in \text{P}(\mathscr{A}, \mathscr{R})$. ■

Proof [Proof of Proposition 13] Consider a pair of sets $(\mathscr{A}, \mathscr{R})$ for which there exists a positive additive coherent set of gambles $\mathscr{D}$, such that $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$. Then the minimal such set is $\uparrow (\mathscr{A} \cup \{0\})$. However, it can be rewritten as:

$$\uparrow (\mathscr{A} \cup \{0\}) = \{g \in \mathscr{L} : PWPC(g) = 1\}$$

where $PWPC(\cdot)$ is a PWP classifier, defined as:

$$PWPC(g) := \begin{cases} 1 & \text{if } \exists f^j \in (\mathscr{A} \cup \{0\}) \text{ s.t. } g \geq f^j, \\ -1 & \text{otherwise.} \end{cases}$$

Therefore, given that $\mathscr{A} \cup T \subseteq \uparrow (\mathscr{A} \cup \{0\}) = \{g : PWPC(g) = 1\}$ and $(\uparrow (\mathscr{A} \cup \{0\}) = \{g : PWPC(g) =$

$1\}) \cap (\mathscr{R} \cup F) = \emptyset$, we have that $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ is PWP separable. Vice versa, consider a PWP separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and a classifier $PWPC(\cdot) \in \mathrm{PWPC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$. Then:

$$\mathscr{D} := \{g : PWPC(g) = 1\}$$

is, by construction, a positive additive coherent set of gambles. Indeed, it clearly satisfies D1, D2, D3**. Further, it is closed because it is a finite intersection of closed sets (respect to the usual topology of $\mathbb{R}^n$) hence, by Proposition 21, it satisfies D4. It satisfies also $\mathscr{D} \supseteq \mathscr{A}$ and $\mathscr{D} \cap \mathscr{R} = \emptyset$ by hypothesis. ■

Proof [Proof of Proposition 15] Consider a PWP separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and a classifier $PWPC(\cdot) \in \mathrm{PWPC}(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\mathscr{F} = \{f^j\}_{j=1}^N$.

Then, a classifier $LC_\rho(\cdot)$ of the type (14), with parameters $\omega^j = f^j$ and $\beta'_j = \begin{bmatrix} 1 \\ \dots \\ 1 \\ -f_1^j \\ \dots \\ -f_n^j \end{bmatrix}$, for all $j = 1, \dots, N$, classifies $\mathscr{A} \cup T$ as 1 and $\mathscr{R} \cup F$ as -1.

Indeed,consider $g \in \mathscr{L}$ and let us define $m := \max_k(\min_l(g_l - f_l^k))$. Then:

$$\sum_{j=1}^{N} (\rho_j(g))^T \beta'_j = \sum_{j=1}^{N} \sum_{i=1}^{n} \mathbb{I}_{\zeta_{ij}}(g)(g_i - f_i^j) = KLm$$

where, for every i, j, ζ_{ij} are the partitions of the type 13 with $\omega^j = f^j$ and where $1 \leq L \leq n$, $1 \leq K \leq N$. Hence, g is classified in the same way by the classifiers $PWPC(\cdot)$ and $LC_\rho(\cdot)$. Therefore, in particular, if $g \in (\mathscr{A} \cup T)$, $m \geq 0$ and hence $LC_\rho(g) = 1$. If instead $g \in (\mathscr{R} \cup F)$ then $m < 0$ and hence $LC_\rho(g) = -1$.

Vice versa, let us consider a P-separable pair $(\mathscr{A} \cup T, \mathscr{R} \cup F)$ and let us suppose the existence of a classifier $LC_\rho(\cdot) \in \mathrm{LC}_P(\mathscr{A} \cup T, \mathscr{R} \cup F)$ with parameters $\{\beta'_j\}_{j=1}^N$ such that $\beta'_{j,i} > 0$, and $\omega_i^j = -\frac{\beta'_{j,i+n}}{\beta'_{j,i}}$ for all $i = 1, .., n$, $j = 1, ..., N$. Let us define $m' := \max_k(\min_l(g_l - (-\frac{\beta'_{k,l+n}}{\beta'_{k,l}})))$. Then, for any $g \in \mathscr{L}$:

$$\sum_{j=1}^{N} (\rho_j(g))^T \beta'_j = \sum_{j=1}^{N} \sum_{i=1}^{n} \mathbb{I}_{\zeta_{i,j}}(g)(\beta'_{j,i} g_i + \beta'_{j,i+n}) =$$

$$\sum_{j=1}^{N} \sum_{i=1}^{n} \beta'_{j,i} \mathbb{I}_{\zeta_{i,j}}(g)(g_i - (-\frac{\beta'_{j,i+n}}{\beta'_{j,i}})), = m' \sum_{j=1}^{K} \sum_{i=1}^{L} \beta'_{j,i}$$

with $1 \leq K \leq N$, $1 \leq L \leq n$. Let us consider a PWP classifier $PWPC(\cdot)$ with parameters $\mathscr{F} = \{f^j\}_{j=1}^N$, such that $f_i^j = -\frac{\beta'_{j,i+n}}{\beta'_{j,i}}$, for all i, j. Then, again, g is classified in the same way by the classifiers $LC_\rho(\cdot)$ and $PWPC(\cdot)$. This is in particular true for $g \in \mathscr{A} \cup T$ and $g \in \mathscr{R} \cup F$. ■