Measuring Dispersion and Concordance of Real-valued Functions With Respect To Fuzzy Measures*

Fernando Reche FRECHE@UAL.ES
María Morales MARIA.MORALES@UAL.ES
Antonio Salmerón ANTONIO.SALMERON@UAL.ES
Department of Mathematics, University of Almería, Spain

We study the problem of defining statistical parameters when the uncertainty is expressed using a fuzzy measure. We extend the concept of monotone expectation [1] in order to define a monotone variance [3]. We also study parameters that allow the joint analysis of two functions defined over the same reference set. The *monotone expectation* [1] of a measurable real-valued function h with respect to a fuzzy measure μ is defined as $E_\mu(h) = \oint h \circ \mu = E_{P_{\mu,h}}(h)$, where $\oint$ denotes the Choquet's integral and $P_{\mu,h}$ is the probability function associated with μ and the ordering induced by h.

Monotone Variance The attempt to define a dispersion parameter as $\mathrm{Var}_\mu(h) = E_\mu[(h - E_\mu(h))^2]$ fails because the distribution associated with μ and the ordering induced by h is not, in general, the same as the one induced by $(h - E_\mu(h))^2$. Thus, we define the *monotone variance* [3] of h with respect to μ as $\mathrm{Var}_\mu(h) = \mathrm{Var}_{P_{\mu,h}}(h)$, i.e. the variance of h computed with respect to the probability measure $P_{\mu,h}$. Note how *monotone moments* can be defined in a similar way.

The Case of Two Functions We introduce the concept of *common expectation* [3] of two real-valued measurable functions as $\psi_\mu(h_1,h_2) = E_\mu[\min\{h_1,h_2\}]$, which allows the definition of the *coefficients of concordance* ρ_1, ρ_2 and ρ_3 between h_1 and h_2 with respect to μ as

$$\rho_1^\mu(h_1,h_2) = \frac{\psi_\mu(h_1,h_2)}{\sqrt{E_\mu(h_1)E_\mu(h_2)}}, \quad \rho_2^\mu(h_1,h_2) = \frac{\psi_\mu(h_1,h_2)}{E_\mu(\max\{h_1,h_2\})}, \quad \rho_3^\mu(h_1,h_2) = \frac{\psi_\mu(h_1,h_2)}{\min\{E_\mu(h_1),E_\mu(h_2)\}}.$$

Example We want to obtain the global grade for two students out of the individual grades they obtained in four different courses $\{x_1,x_2,x_3,x_4\}$. In the final grade we want to reflect if a student shows a good performance in the two scientific courses, $\{x_1,x_2\}$, the humanistic ones, $\{x_3,x_4\}$, or in the combination $\{x_2,x_3\}$, corresponding to a social sciences profile. These criteria are encoded in the fuzzy measure in the table on the left, while the grades obtained by both students (between 0 and 1) in each of the courses are shown in the table on the right.

Reference subsets	μ	Ref. subsets	μ	Ref. subsets	μ
$\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}$	0.2	$\{x_2,x_3\}$	0.5	$\{x_1,x_2,x_3\}$	0.9
$\{x_1,x_2\}$	0.6	$\{x_2,x_4\}$	0.4	$\{x_1,x_2,x_4\}$	0.6
$\{x_1,x_3\}, \{x_1,x_4\}$	0.3	$\{x_3,x_4\}$	0.7	$\{x_1,x_3,x_4\}$	0.7
				$\{x_2,x_3,x_4\}$	0.8

Student	x_1	x_2	x_3	x_4
h_1	0.9	0.8	0.3	0.2
h_2	0.2	0.3	0.8	0.9

The calculation of the respective monotone expectations and variances result in $E_\mu(h_1) = 0.61$, $E_\mu(h_2) = 0.65$, $\mathrm{Var}_\mu(h_1) = 0.0769$, $\mathrm{Var}_\mu(h_2) = 0.0765$, which are quite similar, while the common expectation is $\psi_\mu(h_1,h_2) = 0.25$. The coefficients of concordance are $\rho_1(h_1,h_2) = 0.397, \rho_2(h_1,h_2) = 0.301, \rho_3(h_1,h_2) = 0.410$. Note how they have low values, which is consistent with the data, as in spite of the similar values for the monotone expectation and variance corresponding to both students, they have a clearly different profile, scientific in the case of h_1 and humanistic in the case of h_2.

Product spaces Similar statistical parameters can be defined over product spaces using the product measures introduced in [2]. We extended the concept of monotone expectation to product spaces [3], where, in addition, we have shown how to marginalize the information provided by a function over a product space using the marginal expectations introduced in [2].

References

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