Computing Upper Probabilities Using Global Optimization Algorithms Together With Importance Sampling Techniques

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Let $(X_t)_{t \in T}$ be a family of random variables $X_t : \Omega \to \mathbb{R}^d$ parametrized by t with smooth densities f_t. Then the upper probability $\bar{p} = \max_{t \in T} p(t)$ of an event D is obtained by global optimization where $p(t) = \int \chi_D(x) f_t(x) \, dx$ is the probability function depending on parameter values t and χ_D the indicator function of D. In [3] we introduced a Monte Carlo based estimator

$$\hat{p}_{\omega,s}(t) = \frac{1}{N} \sum_{k=1}^{N} w_{st}(X_s(\omega_k)) \chi_D(X_s(\omega_k))$$

for the probability function p where $w_{st}(x) = f_t(x)/f_s^I(x)$ are the importance sampling weights with a variance reducing importance sampling density f_s^I corresponding to density f_s. This follows from the fact that $p(t) = \int \chi_D(x) f_t(x) \, dx = \int \chi_D(x) \cdot f_t(x)/f_s^I(x) \cdot f_s^I(x) \, dx$. We note that here all sample points $X_s(\omega_k)$, $k = 1, \ldots, N$, are based on the same fixed random numbers $\omega = \{\omega_1, \ldots, \omega_N\}$ on Ω for all $s \in T$, cf. [2]. The case of $s = t$ leads to the estimator $\hat{p}_\omega(t) := \hat{p}_{\omega,t}(t)$ which means classical importance sampling and using different samples for each $t \in T$. For fixed $s \in T$ the smooth function $\hat{p}_{\omega,s}$ is an approximation of $\hat{p}_\omega$ which we used in [3] to compute upper probabilities $\bar{\hat{p}}_{\omega,s}$ while here the focus is on the global optimization itself and how $\hat{p}_\omega$ and $\hat{p}_{\omega,s}$ can be used in the algorithms to compute efficiently estimates of upper probabilities.

The main problem for using global optimization algorithms which are not derivative-free is to compute derivatives of $\hat{p}_\omega$ because it is a step function even if the involved densities are smooth. The idea is here to use the following numerical approximation of the partial derivatives for computing the Jacobian matrix needed in the optimization algorithms:

$$\frac{\partial \hat{p}_\omega}{\partial t_i}(t) \approx \frac{\hat{p}_{\omega,t}(t + h^{(i)}) - \hat{p}_{\omega,t}(t)}{h^{(i)}} \quad \text{with elements} \quad h_k^{(i)} = 10^{-8} \, \delta_{ik} \quad \text{of the vector } h^{(i)} \text{ of step sizes.}$$

We note that in this numerical approximation the computation of both estimators in the formula are based on the same sample which means that we locally differentiate a smooth function.

A second problem is that the evaluation of χ_D may be very time consuming, e.g. finite element computations for upper probabilities of failure of an engineering structure, see [1]. In [3] we have seen that $\hat{p}_{\omega,s}(t)$ may be a bad approximation of $\hat{p}_\omega(t)$ for t far from parameter value s on which the sample is based and therefore not suitable to replace $\hat{p}_\omega$ by one single function $\hat{p}_{\omega,s}$. Running an optimization algorithm leads to a sequence $t^{(1)}, t^{(2)}, \ldots, t^{(i)}, \ldots, t^{(j)}, \ldots$ of parameter values. Assuming now that in step i the estimator for the probability is computed in the more exact way using $\hat{p}_{\omega,t^{(i)}}(t^{(i)})$ based on the sample corresponding to $t^{(i)}$ itself then the idea is to use an approximation $\hat{p}_{\omega,t^{(i)}}(t^{(j)})$ at $t^{(j)}$ if $\|t^{(j)} - t^{(i)}\| < \mathsf{TOL}$ with TOL small enough obtained in an adaptive way. This saves us N evaluations $\chi_D(X_{t^{(j)}}(\omega_k))$ because no new sample points are needed.

In the poster presentation we will apply these two methods for the algorithms in the MATLAB global optimization toolbox including examples.

References

[1] Thomas Fetz. Efficient computation of upper probabilities of failure. In Christian Bucher, Bruce R. Ellingwood, and Dan M. Frangopol, editors, *12th International Conference on Structural Safety & Reliability*, pages 493–502, 2017.

[2] Matthias C. M. Troffaes. A note on imprecise Monte Carlo over credal sets via importance sampling. In Alessandro Antonucci, Giorgio Corani, Inés Couso, and Sébastien Destercke, editors, *Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications*, volume 62 of *Proceedings of Machine Learning Research*, pages 325–332. PMLR, July 2017. URL http://proceedings.mlr.press/v62/troffaes17a.html.

[3] Matthias C. M. Troffaes, Thomas Fetz, and Michael Oberguggenberger. Iterative importance sampling for estimating expectation bounds under partial probability specifications. In Marco De Angelis, editor, *Proc. of the 8th International Workshop on Reliable Engineering Computing*, pages 147–154, Liverpool, 2018. Liverpool University Press.